Eidgenössische Technische Hochschule Zürich
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Morse-Bott Homology

Master Thesis

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March 4, 2025

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Abstract

Morse-Bott functions generalize Morse functions, permitting the critical points of a function to form submanifolds, rather than being isolated. In fact, many natural maps are Morse-Bott, thanks to symmetries. In this thesis, we develop the Morse-Bott homology of a Morse-Bott function and show that this homology is isomorphic to the singular homology. We apply this method to make computations towards the homology of the special orthogonal groups, $SO(n)$.

Acknowledgements

I would like to express my gratitude towards my advisors, Prof. Paul Biran and Dr. Ipsita Datta for their support in the process of writing this thesis. In particular, I am deeply grateful to Dr. Ipsita Datta for her continuous help and valueable feedback during the writing process.

Contents

Chapter 1

Introduction

Morse theory provides a way to compute the singular homology of a smooth manifold using a smooth function with isolated critical points, called Morse function. Morse-Bott functions is a generalization of Morse functions, permitting the critical points to form submanifolds. In this thesis, we develop a way to compute the singular homology of a manifold using a Morse-Bott function, following Banyaga and Hurtubise's construction in [1].

We then use the developed theory to compute the homology of the special orthogonal group, $SO(n)$. We define the Morse-Bott function $f : SO(n) \to \mathbb{R}$ defined by $f(X) = X_{nn}$ (the lower-right coordinate). This function has two critical submanifolds, each diffeomorphic to $SO(n-1)$. In particular, we show that the short sequence

$$0 \to H_k(SO(2n-1)) \to H_k(SO(2n)) \to H_{k-2n+1}(SO(2n-1)) \to 0.$$

is exact (Theorem 3.18). In addition, we show a recursive formula for the mod 2 homology of $SO(n)$ (Theorem 3.23):

$$H_k(SO(n); \mathbb{Z}_2) \cong H_k(SO(n-1)) \oplus H_{k-n+1}(SO(n-1)).$$

Overview

Morse-Bott Homology

Let B be a topological space. For every integer $k \geq 0$, the k^{th}-homology group $H_k(B)$ is an abelian group that is a topological invariant of B. Roughly speaking, the rank of $H_k(B)$ measures the number of k-dimensional holes.

There are multiple ways to compute the homology groups of a topological space B. In all cases, the process is similar: For every integer $k \geq 0$, we associate an abelian group C_k, which is derived from B in some way. Then, for every k, we define a linear map $\partial : C_k \to C_{k-1}$ such that

$$\partial \circ \partial : C_k \to C_{k-2}$$

is the zero map for every $k \geq 0$. This pair $(C_\bullet, \partial)$ is called a *chain complex*. The *homology groups* of the chain complex $(C_\bullet, \partial)$ are defined as

$$H_k(C_\bullet, \partial) := \frac{\ker(\partial : C_k \to C_{k-1})}{\operatorname{Im}(\partial : C_k \to C_{k-1})}.$$

The most generic homology theory is the singular homology. In this case, we define $\Delta^k \in \mathbb{R}^{k+1}$, the k-dimensional simplex, to be k-dimensional polygon whose vertices are

$$v_0 = (1, 0, \ldots, 0),\ v_1 = (0, 1, 0, \ldots, 0), \ldots, v_{k+1} = (0, \ldots, 0, 1)$$

and call a continuous map $\sigma : \Delta^k \to B$ a *singular chain of degree k*. We define $S_k(B)$ to be the free abelian group generated by all singular chains of degree k. The boundary operator $\partial : S_k(B) \to S_{k-1}(B)$ is defined on generators by

$$\partial(\sigma) = \sum_{i=1}^{k+1} (-1)^i \, \sigma|_{[v_1, \ldots, \widehat{v}_i, \ldots, v_{k+1}]}$$

where $[v_1, \ldots, \widehat{v}_i, \ldots, v_{k+1}]$ is the face of Δ^k containing all vertices except v_i. For more details about singular homology, see Chapter 2 of [2].

If M is a compact orientable manifold, one can use Morse homology to compute the homology groups of M. A smooth function $f : M \to \mathbb{R}$ is called *Morse* if in local coordinates, the Hessian matrix

$$\operatorname{Hess}_x(f) = \left(\frac{d^2 f}{dx_i dx_j} \right)_{ij} (x)$$

of every critical point

$$x \in \operatorname{Crit}(f) := \{x \in M \mid df_x \equiv 0\}$$

has full rank. In this case, we define the index of $x \in \operatorname{Crit}(f)$ to be the number of negative eigenvalues of $\operatorname{Hess}_x(f)$ and set $\hat{B}_k$ to be the set of all critical points of index k. Given a metric g on M, we define $\varphi_t(x)$ to be the flow of $-\nabla f$. Then, we define $C_k(f)$ to be the free abelian group generated by $\hat{B}_k$, and we define $\partial : C_k(f) \to C_{k-1}(f)$ on generators by

$$\partial(p) = \sum_{q \in \hat{B}_{k-1}} n(p, q) q$$

where, informally speaking, $n(p, q)$ counts the signed number of flow lines from p to q. The Morse homology theorem states that $(C_k(f), \partial)$ is a chain complex whose homology is the same as the singular homology and, in particular, does not depend on the function f. Some references for Morse homology include [3], [4], [5] and [6].

The Kupka-Smale theorem [7, Theorem 3.1] states that the set of Morse functions is dense in the space of all smooth functions. In particular, every smooth function can be perturbed to a Morse function. However, in some cases, a natural function might not be Morse, and some properties of the functions might be lost in perturbation. For example, consider the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates induced from $\mathbb{R}^2$. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be defined by $f(x,y) = \cos 2\pi x$. Then the critical points of f are

$$\mathrm{Crit}(f) = \left\{ (x,y) \in \mathbb{T}^2 \mid x = 0, \frac{1}{2} \right\}$$

which is not an isolated set. This function is invariant under the $\mathbb{R}$-action $t \cdot (x,y) \mapsto (x, y+t)$, but no Morse function is invariant under this action.

Another example, which is discussed in detail in Chapter 3 of this thesis and was taken from Section 5 of [8], is the function $f : SO(n) \to \mathbb{R}$, defined by $f(X) = X_{nn}$ (the lower-right coordinate of X). In this case, $\mathrm{Crit}(f)$ consists of two copies of $SO(n-1)$.

A generalization of a Morse function, called a Morse-Bott function, is a function $f : M \to \mathbb{R}$ such that $\mathrm{Crit}(f)$ is not a discrete set, but rather a finite union of submanifolds of M, and the non-degeneracy condition is replaced by a similar condition (Definition 2.1).

There are few different approaches for Morse-Bott homology, all described in detail in [9]. In addition to perturbing the function, cascade lines can also be used to compute the Morse-Bott homology of f. More details on cascades can be found in [10].

The approach we have chosen to focus on this thesis is rather different. In this approach, a chain complex $(C_\bullet(\hat{B}_i), \partial_0)$ is assigned to each index of critical points, and a map

$$\partial_j : C_k(\hat{B}_i) \to C_{k+j-1}(\hat{B}_{i-j})$$

is defined for every $j = 1, \ldots, m$ and $k = 0, \ldots, m$. The chain complex $(C_k(f), \partial)$ is then defined by

$$C_k(f) := \bigoplus_{i=0}^{k} C_i(\hat{B}_{k-i})$$

and $\partial = \bigoplus_{i=0}^{k} \partial_j : C_k(f) \to C_{k-1}(f)$. The complex can be pictured as follows:

$$
\begin{tikzcd}
\ddots & \oplus & & & & \\
\dots & C_1(\hat{B}_2) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] \arrow[ddddr, "\partial_2"'] & C_0(\hat{B}_2) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] \arrow[ddddr, "\partial_2"'] & 0 & & \\
 & \oplus & \oplus & & & \\
\dots & C_2(\hat{B}_1) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & C_1(\hat{B}_1) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & C_0(\hat{B}_1) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & 0 & \\
 & \oplus & \oplus & \oplus & & \\
\dots & C_3(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[d, equal] & C_2(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[d, equal] & C_1(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[d, equal] & C_0(\hat{B}_0) \arrow[r] \arrow[d, equal] & 0 \\
\dots & C_3(f) \arrow[r, "\partial"] & C_2(f) \arrow[r, "\partial"] & C_1(f) \arrow[r, "\partial"] & C_0(f) \arrow[r, "\partial"] & 0
\end{tikzcd}
$$

An early development of Morse-Bott theory can be found in [11], where Austin and Braam used differential forms to construct Morse-Bott cohomology, and showed that it is isomorphic to the De Rham cohomology. Other approaches can also be found in [12], [13], [14], [15], [1], and [16]. In this thesis, we focus on the approach of Banyaga and Hurtubise in [1].

In [1], Banyaga and Hurtubise define the singular N-cube homology, where C_k is defined to be the set of all k-faces of I^N (for some N large enough), and the group $S_k(B)$ is generated by all continuous maps $\sigma : P \to B$ and $P \in C_k$. Then, the subgroup $D_k(B) \subset S_k(B)$ of *degenerate singular N-cube chains* is defined by some degeneracy conditions, described in Definition 2.10. The chain complex $(S_\bullet(B)/D_\bullet(B), \partial)$ is called the *singular N-cube chain complex* and its homology groups are shown to be isomorphic to the singular homology.

For the boundary maps $\partial_j : C_p(\hat{B}_i) \to C_{p+j-1}(\hat{B_{i-j}})$, a construction named *fibered product* is utilized. Given maps $\sigma_i : P_i \to B$ ($i = 1, 2$), the fibered product of σ_1 and σ_2 is defined as

$$P_1 \times_{\sigma_1, B, \sigma_2} P_2 := \{(x, y) \in P_1 \times P_2 \mid \sigma_1(x) = \sigma_2(y)\}.$$

If $\overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j})$ is the space of all piecewise gradient flow lines from $\hat{B}_i$ to $\hat{B}_{i-j}$ and

$$e_- : \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \to \hat{B}_i, \; e_+ : \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \to \hat{B}_{i-j}$$

are the beginning and endpoints maps of the piecewise gradient flow lines, one can describe

$$P \times_{\sigma_P, \hat{B}_i, e_-} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) := P \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j})$$

as the space of all possible ways to get from P to $\hat{B}_{i-j}$ through $\sigma_P : P \to \hat{B}_i$. Therefore, if σ_P is a smooth singular chain, $\partial_j(\sigma_P)$ is defined to be the singular chain

$$\partial_j(\sigma_P) : P \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \xrightarrow{e_+} \hat{B}_{i-j}.$$

Then, $S_p^\infty(\hat{B}_i)$ is defined to be the free abelian group generated by smooth maps $\sigma_P : P \to \hat{B}_i$ where P is a p-face of I^n, but $\partial_j(\sigma_P)$ is also added as generators of $S_{p+j-1}^\infty(\hat{B}_{i-j})$, so

$$\partial_j : S_k^\infty(\hat{B}_i) \to S_{p+j-1}^\infty(\hat{B}_{i-j})$$

is well-defined.

Afterwards, the subgroup of degenerate singular chains $D_k\infty(\hat{B}_i)$ is defined by some degeneracy conditions, including the conditions for degenerate singular N-cube chains, but also for identifying $\partial_j(\sigma_P)$ with a smooth singular N-cube chain. The chain complex $(C_k(\hat{B}_i), \partial_0)$ is defined by

$$C_k(\hat{B}_i) := S_k^\infty(\hat{B}_i)/D_k^\infty(\hat{B}_i)$$

and

$$\partial_0 : S_k^\infty(\hat{B}_i) \to S_{k-1}^\infty(\hat{B}_i)$$

is $(-1)^k$ times the boundary operator defined on smooth singular N-cube chains. Then, it is shown that $(C_k(f), \partial)$ is a chain complex whose homology does not depend on f (Theorem 2.44).

If $f : M \to \mathbb{R}$ is a constant function (Example 2.40), then $\hat{B}_0 = M$ and $\hat{B}_i = \emptyset$ for $i \geq 0$. The chain complex $(C_k(f), \partial)$ is then the smooth version of the singular N-cube chain complex, whose homology is isomorphic to the singular homology.

On the other hand, if $f : M \to \mathbb{R}$ is a Morse-Smale function (Example 2.41), then the chain complex $(C_k(f), \partial)$ is the Morse-Smale chain complex, and therefore the homology of $(C_k(f), \partial)$ is isomorphic to the Morse homology. Hence, since the homology is independent of the function, Theorem 2.44 gives another proof of the Morse homology theorem.

Orthogonal Groups

The *orthogonal group* consists of all matrices of size $n \times n$ whose rows are unit vectors, orthogonal to each other. Formally,

$$O(n) := \left\{X \in M_{n\times n}(\mathbb{R}) \mid XX^t = I_n\right\}$$

where I_n is the identity matrix. The *special orthogonal group* is the subgroup of all orthogonal matrices with determinant 1. That is,

$$SO(n) := \{X \in O(n) \mid \det X = 1\}.$$

$O(n)$ has two connected components, each homeomorphic to $SO(n)$. Thus, computing the homology groups of $SO(n)$ gives also the homology groups of $O(n)$ by

$$H_\bullet(O(n)) = H_\bullet(SO(n)) \oplus H_\bullet(SO(n)).$$

In Chapter 3 we compute the homology groups of $SO(n)$. We apply the system defined in Chapter 2 with methods from [8]. As in [8], we define the Morse-Bott function $f : M \to \mathbb{R}$, $f(X) = X_{nn}$ which takes the lower-right coordinate of X. The function has two critical submanifolds with indices 0 and $n-1$ denoted by F_0 and F_{n-1} respectively, each diffeomorphic to $SO(n-1)$. Therefore, one can compute the homology groups of $SO(n)$ recursively using the Morse-Bott complex of f.

The idea for the computations is inspired by Section 5 of [8]. We first show that the space $\mathcal{M}(F_{n-1}, F_0)$ is diffeomorphic to $SO(n-1) \times S^{n-2}$. Using this identification, the beginning and endpoint maps e_-, e_+ satisfy

$$e_-(X, v) = e_-(X, -v), \; e_+(X, v) = e_+(X, -v)$$

for $X \in SO(n-1)$ and $v \in S^{n-2}$. Then, for every cycle $\sigma \in C_p(F_{n-1})$, we find a p-dimensional CW-complex P', a map

$$\varphi : P' \times S^{n-2} \xrightarrow{(x,v)\mapsto(x,[v])} P' \times \mathbb{R}P^{n-2} \to F_{n-1}$$

and a cycle $\sigma' \in S_{p+n-2}\left(P' \times S^{n-2}\right)$ such that $\varphi_*(\sigma') = \partial_{n-1}(\sigma)$. We then show that

$$\varphi_* : H_{p+n-2}(P' \times S^{n-2}) \to H_{p+n-2}(F_0)$$

is the zero map if n is even (or with $\mathbb{Z}_2$ coefficients) and therefore, $(\partial_{n-1})_* : H_p(F_{n-1}) \to H_{p+n-2}(F_0)$ is the zero map on homology. Using this fact, we get our results.

Our main result is Theorem 3.18, which gives the short exact sequence

$$0 \to H_k(SO(2n-1)) \to H_k(SO(2n)) \to H_{k-2n+1}(SO(2n-1)) \to 0.$$

Although it might be possible to deduce this short exact sequence from other computations of the homology groups of $SO(n)$, such as in [17] and [2, Chapter 3D], this is the first time the short exact sequence is written explicitly.

In addition, we compute the mod 2 homology groups of $SO(n)$. Theorem 3.23 states that

$$H_k(SO(n); \mathbb{Z}_2) \cong H_k(SO(n-1); \mathbb{Z}_2) \oplus H_{k-n+1}(SO(n-1); \mathbb{Z}_2).$$

This theorem is already known. Our proof is very similar to the one in [8, Section 5], where they used the same Morse-Bott function but with different Morse-Bott system. Hatcher [2, Theorem 3D.1] gives another proof using cell structures, and [18, Theorem 3] provides a proof using Morse functions.

Structure Of The Thesis

In Chapter 2 we construct Banyaga and Hurtubise's Morse-Bott system, as described in [1]. We start in Section 2.1, in which we define a Morse-Bott function and state the required transversality conditions. These requirements are fulfilled in most cases, including the constant function, Morse-Smale functions, and the function discussed in Section 3.3.

In Section 2.2 we construct the framework we later use for the boundary operator ∂_0. We start in Subsection 2.2.1, where we formalize a singular topology framework, of which the singular homology is a special case. In Subsections 2.2.2 and 2.2.3, we define the N-cube singular homology using the singular homology framework. In Subsections 2.2.4 and 2.2.5, we extend the boundary operator to fibered products and compactificated moduli spaces as well.

We define our chain complex in Section 2.3. The chain complex consists of smooth maps from spaces defined in Section 2.2. We also add degeneracy conditions in Subsection 2.3.4 and define the *Morse-Bott homology* in Subsection 2.3.5. In Section 2.4 we prove that the homology of the chain complex is independent on the choice of the function, and in particular isomorphic to the singular homology. In the last section in Chapter 2, we define the Morse-Bott homology with coefficients in a field, and show that it is isomorphic to the singular homology with the same coefficients.

In Chapter 3, we use the Morse-Bott-Smale chain complex defined in Chapter 2 to compute the homology groups of $SO(n)$. To our knowledge, this is the first application of this Morse-Bott theory. Section 3.1 is dedicated for linear Morse-Bott functions on $SO(n)$, of which our function $f(X) = X_{nn}$ is a special case. In Section 3.2 we define the mapping cone, and show that the chain complex of simple Morse-Bott functions (that is, only two critical submanifolds) can be described as a mapping cone. In Section 3.3 we put everything together to compute the homology groups of $SO(n)$.

Further Directions

In this thesis we computed the homology groups of $SO(2n)$ using the homology groups of $SO(2n-1)$. However, the odd case is not discussed in this thesis at all. Computing the homology groups of $SO(2n-1)$ is a natural continuation of this thesis. In addition, the framework and ideas we developed might be applicable for other cases as well, like the unitary group $U(n)$.

Another possible direction is trying to soften the transversality condition. Currently, there are Morse-Bott functions (Example 2.7) which cannot meet the required transversality conditions under any Riemannian metric.

Chapter 2

Morse-Bott Functions

2.1 Morse-Bott Functions and Transversality

In this section we define a Morse-Bott function. Then, we define the stable and unstable manifolds of a critical submanifold, and state the transversality conditions required for these manifolds. The stable and unstable manifolds are key parts of defining the flow lines and boundary maps later. The section follows Section 3 of [1] closely.

Let (M, g) be an m-dimensional compact Riemannian manifold and $f \in C^\infty(M)$. We call $\mathrm{Crit}(f) := \{p \in M | \ df_p \equiv 0\}$ the set of *critical points* of f. Assume now that $\mathrm{Crit}(f) = \bigsqcup_{i=1}^n B_i$ is a finite union of connected manifolds B_i of M, called *critical submanifolds*. Let B be a critical submanifold. Note that $df|_B \equiv 0$ and B is connected, so f is constant on B.

For every $p \in B$, the tangent space T_pM splits to the tangent space in B and $\nu_p(B)$ (the normal bundle of B in M at p). i.e.

$$T_pM = T_pB \oplus \nu_p(B).$$

The normal bundle of B is defined as

$$\nu_*(B) = \bigcup_{p \in B} \nu_p(B)$$

and is a vector bundle of rank $(m - \dim B)$ of B [19, Proposition 2.16].

The *Hessian* of f at $p \in M$ is the symmetric bilinear form

$$\mathrm{Hess}_p(f) : T_pM \times T_pM \to \mathbb{R}$$

that is defined by $\mathrm{Hess}_p(f)(V, W) = V \cdot \left(\tilde{W} \cdot f\right)$, where $\tilde{W}$ is any extension of W to M. If $V \in T_pB$, then

$$\mathrm{Hess}_p(f)(W, V) = \mathrm{Hess}_p(f)(V, W) = V \cdot \left(\tilde{W} \cdot f\right)\big|_p = 0$$

because $\tilde{W} \cdot f\big|_q = 0$ for every $q \in B$. Therefore $\mathrm{Hess}_p(f)$ reduces to a symmetric bilinear form:

$$\mathrm{Hess}_p^\nu(f) : \nu_p(B) \times \nu_p(B) \to \mathbb{R},$$

which is called the *normal Hessian* of f at p.

Definition 2.1 $f : M \to \mathbb{R}$ is called *Morse-Bott* if $\mathrm{Crit}(f)$ is a disjoint union of connected submanifolds of M and Hess_p^ν is non-degenerate for every critical submanifold $B \subset \mathrm{Crit}(f)$ and $p \in B$.

We can generalize Morse lemma to critical submanifolds:

Theorem 2.2 (Morse-Bott Lemma) *[20, Theorem 2] Let $f : M \to \mathbb{R}$ be Morse-Bott. Then for every critical submanifold $B \subset \mathrm{Crit}(f)$ and $p \in B$, there is a local chart (U, φ) around p such that*

$$f(x) = f(B) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_{i+j}^2$$

where $i + j = \dim \nu_p(B)$. Such a chart is called a Morse-Bott chart of p.

We call $\lambda_p := i$ the *index* of p and denote $\lambda_p^* := j$. The index λ_p can be also described as the number of negative eigenvalues of $\mathrm{Hess}_p(f)$. Since λ_p and λ_p^* are constant on a Morse-Bott chart and B is connected, λ_p and λ_p^* are constant on B. Hence, we can define indices for B as

$$\lambda_B := \lambda_p, \quad \lambda_B^* = \lambda_p^*$$

for some $p \in B$. In addition, Morse-Bott Lemma induces a local splitting of the normal bundle $\nu_*(B)$ to

$$\nu_*(B) = \nu_*^-(B) \oplus \nu_*^+(B)$$

where $\nu_*^-(B)$ and $\nu_*^+(B)$ are defined in a Morse-Bott chart U as

$$\nu_*^-(B) = \mathrm{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_i}\right\}, \; \nu_*^-(B) = \mathrm{span}\left\{\frac{\partial}{\partial x_{i+1}}, \ldots, \frac{\partial}{\partial x_{i+j}}\right\}.$$

Definition 2.3 Let $f : M \to \mathbb{R}$ be Morse-Bott. A *pseudo-gradient* on M is a vector field $X \in \Gamma(TM)$ satisfying:

- For every $p \in \mathrm{Crit}(f)$ there is a Morse-Bott chart U around p so that $X = -\nabla f$.
- For every $q \in M \setminus \mathrm{Crit}(f)$, $(X \cdot f)\big|_q < 0$.

Observe that for every $p \in M$

$$(-\nabla f)(p) \cdot f = df_p(-\nabla f) = \langle \nabla f(p), (-\nabla f)(p) \rangle = -\left|\nabla f(p)\right|^2 \leq 0$$

and $(-\nabla f)(p) \cdot f = 0$ if and only if $\nabla f(p) = 0$. That is, $p \in \mathrm{Crit}(f)$. Therefore, $-\nabla f$ is a pseudo-gradient.

Definition 2.4 Fix a pseudo-gradient X. Let φ_t be the flow of X (note that φ_t is defined for all time because M is compact). We define the *stable* and *unstable manifolds* of $p \in \mathrm{Crit}(f)$ the same way as for Morse functions:

$$W^s(p) = \left\{ x \in M | \lim_{t\to\infty} \varphi_t(x) = p \right\}$$

$$W^u(p) = \left\{ x \in M | \lim_{t\to-\infty} \varphi_t(x) = p \right\}$$

And for a critical submanifold $B \in \mathrm{Crit}(f)$:

$$W^s(B) = \bigcup_{p\in B} W^s(p)$$

$$W^u(B) = \bigcup_{p\in B} W^u(p)$$

Theorem 2.5 *[11, Proposition 3.2] There are smooth injective immersions*

$$E^+ : \nu_*^+(B) \to M$$

and

$$E^- : \nu_*^-(B) \to M$$

with $W^s(B)$ and $W^u(B)$ as their images.
In addition, there are smooth endpoint maps

$$e_+ : W^s(B) \to B \text{ and } e_- : W^u(B) \to B$$

given by $e_\pm(x) = \lim_{t\to\pm\infty} \varphi_t(x)$ which have the structure of a locally trivial fiber bundle when restricted to a neighborhood of B.

We say that the pair (f, X) consists of a Morse-Bott function $f : M \to \mathbb{R}$ and a pseudo-gradient X is *Morse-Bott-Smale* if $W^u(p)$ and $W^s(B')$ intersect transversely for every two critical submanifolds $B, B' \subset \mathrm{Crit}(f)$ and $p \in B$. We say that f is Morse-Bott-Smale if $(f, -\nabla f)$ is a Morse-Bott-Smale pair.

If f is a Morse function, then $-\nabla f$ can always be approximated by a pseudo-gradient X such that (f, X) is a Morse-Smale pair [21, Theorem A]. However, there exist Morse-Bott functions that are not Morse-Bott-Smale with respect to any Riemannian metric, or any gradient-like vector field (see Example 2.7).

If $f : M \to \mathbb{R}$ is Morse-Bott-Smale, then the space

$$W(B, B') := W^u(B) \pitchfork W^s(B')$$

is a submanifold, as it is a transverse intersection of two manifolds. We denote $b = \dim B$, $b' = \dim B'$. Then

$$\dim W^u(B) = b + \lambda_B,$$

$$\dim W^s(B') = b' + \lambda^*_{B'} = m - \lambda_{B'},$$

and if $W(B,B') \neq \emptyset$, then $\dim W(B,B') = \lambda_B - \lambda_{B'} + b$.

A Morse-Bott function is called *weakly self-indexing* if for every pair of critical submanifolds $B \neq B'$ such that $\lambda_B \leq \lambda_{B'}$, $W(B,B') = \emptyset$, i.e. the index is strictly decreasing along flow lines.

Lemma 2.6 *Let $f : M \to \mathbb{R}$ be Morse-Bott-Smale. Then f is weakly self-indexing [1, Lemma 3.6].*

Proof Let B, B' be critical submanifolds such that $W(B,B') \neq \emptyset$. Then there is $x \in B$, $W(x,B') := W^u(x) \cap W^s(B') \neq \emptyset$. By transversality,

$$\dim W(x,B') = \lambda_B + m - \lambda'_B - m = \lambda_B - \lambda'_B.$$

Take $y \in W(x,B')$. Then for every $t \in \mathbb{R}$, $\varphi_t(y) \in W(x,B')$ and hence $\dim W(x,B') \geq 1$ since $t \mapsto \varphi_t(y)$ is an injective immersion of $\mathbb{R}$ into $W(x,B')$. Thus,

$$\lambda_B - \lambda'_B = \dim W(x,B') \geq 1$$

so f is weakly self-indexing. □

The next example is adapted from [15, Remark 2.4].

Example 2.7 (Morse-Bott function that is not Morse-Bott-Smale) Let $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ be the two-dimensional torus with the parametrization induced from $\mathbb{R}^2$ and let $f : M \to \mathbb{R}$ be

$$f(x,y) = -(2 + \cos 2x)(1 + \cos y).$$

Then f has $B_{top} := \{(x,\pi) | 0 \leq x < 2\pi\}$ as a critical submanifold of index 1, which means B_{top} consists of local maxima, since $\dim B_{top} = 1$; two saddle points, $(\pi/2, 0)$ and $(3\pi/2, 0)$, and two local minima, $(0,0)$ and $(\pi, 0)$.

Let $p = (\pi/2, 0)$. Since $\dim W^s(p) = 1$, there is an $x \neq p$ in $W^s(p)$. Take $q := \lim_{t\to-\infty} \varphi_t(x) \in B_{top}$. Then $W^u(q) \cap W^s(p) = \{\varphi_t(y) \mid t \in \mathbb{R}\} \neq \emptyset$ and in particular,

$$W^u(B_{top}) \cap W^s(p) \neq \emptyset.$$

But $\lambda_p = \lambda_{B_{top}}$ and therefore f is not weakly self-indexing, and hence, not Morse-Bott-Smale.

2.2 Topological Chains and Fibered Products

In this section, we begin by introducing an abstract singular chain complex, and define the singular N-cube chain complex, whose homology is isomorphic to the singular homology. Next, we define fibered products, and assign a boundary operator for fibered products as well. In the final part of this section, we define the compactificated moduli spaces, and also define a boundary operator. The chain complex is defined as the smooth singular N-cube chain complex, enriched by smooth maps from fibered products of faces with compactificated moduli spaces. This section follows Section 4 of [1].

2.2.1 Some General Definitions

For every integer $p \geq 0$ and a fixed set C_p of topological spaces, we define S_p to be the free abelian group generated by C_p. If $p < 0$ or $C_p = \emptyset$, then we define $S_p = 0$. We call the elements of S_p *abstract topological chains of degree p*.

In our case, B will be a manifold and C_p will contain manifolds with corners of dimension p. However, this is not required by the definition.

Definition 2.8 A *boundary operator* on a collection of abelian groups $\{S_p\}_{p\geq 0}$ is a collection of homomorphisms $\{\partial_p : S_p \to S_{p-1}\}_{p\geq 0}$ so that

$$\partial_{p-1} \circ \partial_p : S_p \to S_{p-2}$$

is the zero homomorphism for every p. To simplify the notation, we will usually omit the index from ∂_p.

2.2.2 Abstract N-Cubes Chains

Let M be a smooth manifold. Fix some large $N > \dim M$ and define

$$I^N := [0,1]^N.$$

Let C_p be the set of p-faces of I^N. The *boundary map* $d_p : S_p \to S_{p-1}$ is defined on generators $P \in C_p$ by

$$d_p(P) = \sum_{j=0}^{p} (-1)^j \left[P|_{x_j=1} - P|_{x_j=0} \right]$$

where x_j denotes the j^{th} coordinate of P, and extended linearly.

Lemma 2.9 *The collection $\{d_p : S_p \to S_{p-1}\}_{p\geq 0}$ is a boundary operator, that is, $d_{p-1} \circ d_p : S_p \to S_{p-2}$ is the zero map for every p.*

Proof It is sufficient to show for generators. Let $P \in C_p$. Then,

$$d_{p-1} \circ d_p(P) = \sum_{j=0}^{p} (-1)^j \left[d_p\ P|_{x_j=1} - d_p\ P|_{x_j=0} \right]$$

$$
\begin{aligned}
&= \sum_{j=0}^{p}(-1)^j \Bigg[\sum_{i=0}^{p-1}(-1)^i \Big[\Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=1} - \Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=0} \\
&\qquad\qquad - \Big(P|_{x_j=0}\Big)\Big|_{\tilde{x}_i=1} + \Big(P|_{x_j=0}\Big)\Big|_{\tilde{x}_i=0} \Big]\Bigg] \\
&= \sum_{j=0}^{p} \Bigg[\sum_{i=0}^{p-1}(-1)^{i+j} \Big[\Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=1} - \Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=0} \\
&\qquad\qquad - \Big(P|_{x_j=0}\Big)\Big|_{\tilde{x}_i=1} + \Big(P|_{x_j=0}\Big)\Big|_{\tilde{x}_i=0} \Big]\Bigg].
\end{aligned}
$$

Now, let us expand the first term of the last line:

$$
\begin{aligned}
\sum_{j=1}^{p}\sum_{i=0}^{p-1}(-1)^{i+j} \Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=1} &= \sum_{j=1}^{p}\sum_{i=0}^{j-1}(-1)^{i+j} \Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=1} \\
&\qquad + (-1)^{i+j+1} \Big(P|_{x_i=1}\Big)\Big|_{\tilde{x}_{j+1}=1}.
\end{aligned}
$$

If $i < j$, then

$$
\Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=1} = \Big(P|_{x_i=1}\Big)\Big|_{\tilde{x}_{j+1}=1}.
$$

Therefore,

$$
\sum_{j=1}^{p}\sum_{i=0}^{p-1}(-1)^{i+j} \Big(P|_{x_j=1}\Big)\Big|_{\tilde{x}_i=1} = 0.
$$

The remaining terms of $d_{p-1} \circ d_p(P)$ can be shown to cancel similarly. □

2.2.3 Singular N-Cube Chains

Let B be a topological space. A *singular C_p-space of B* is a continuous map $\sigma : P \to B$ for some $P \in C_p$. The set of all singular C_p-spaces of B is denoted by $C_p(B)$. The *singular C_p-chain group*, $S_p(B)$, is defined to be the free abelian group generated by $C_p(B)$. The elements of $S_p(B)$ are called *singular topological chains of degree p*.

Let $P \in C_p$ and write

$$
d_p(P) = \sum_j n_j(P) P_j
$$

where $P_j \in C_{p-1}$. For a singular C_p-space $\sigma_P : P \to B$, we define $\partial_p : S_p(B) \to S_{p-1}(B)$ by $\partial_p(\sigma_P) = \sum_j n_j(P)\, \sigma_P|_{P_j}$. Then

$$
\partial_{p-1} \circ \partial_p : S_p(B) \to S_{p-2}(B)
$$

is also the zero map. Thus, we can define the *homology groups* of $(S_\bullet(B), \partial_\bullet)$ by

$$H_p(S_\bullet(B), \partial_\bullet) := \ker \partial_p / \mathrm{Im}\partial_{p+1}.$$

Definition 2.10 Let σ_P, σ_Q be singular C_p-spaces and denote $\partial_p(Q) = \sum_j n_j Q_j$. For a fixed continuous map $\alpha : P \to Q$, we define

$$\partial_p(\sigma_Q) \circ \alpha = \sum_j n_j \left.(\sigma_Q \circ \alpha)\right|_{\alpha^{-1}(Q_j)}.$$

The subgroup of *degenerate singular N-cubes chains* $D_p(B) \subset S_p(B)$ is the subgroup generated by the following elements:

1. If $\alpha : P \to Q$ is an orientation preserving homeomorphism such that $\sigma_P = \sigma_Q \circ \alpha$ and $\partial_p(\sigma_P) = \partial_p(\sigma_Q) \circ \alpha$, then $\sigma_P - \sigma_Q \in D_p(B)$.
2. If σ_P does not depend on some free coordinate of P (that is, there is $1 \leq j \leq p$ such that σ_P does not depend on x_j), then $\sigma_P \in D_p(B)$.

Theorem 2.11 (Singular N-cube Chain Theorem) *[1, Theorem 4.4]*

The boundary operator $\partial_p : S_p(B) \to S_{p-1}(B)$ induces a homomorphism

$$\partial_p : S_p(B)/D_p(B) \to S_{p-1}(B)/D_{p-1}(B)$$

and

$$H_p\left(S_\bullet(B)/D_\bullet(B), \partial_\bullet\right) = H_p(B; \mathbb{Z}).$$

for all $p < N$.

Let M be a smooth manifold. We define the group of *smooth singular C_p-chain group*, $S_p^\infty(M)$ as the subgroup of $S_p(M)$ generated by smooth singular C_p-spaces $\sigma : P \to M$. Similarly, $D_p^\infty(M)$ is defined to be the subgroup of $S_p^\infty(M)$ generated by the conditions in Definition 2.10.

Remark 2.12 *The proof of Theorem 2.11 can be applied verbatim to show that*

$$H_\bullet\left(S_\bullet^\infty(M)/D_\bullet^\infty(M), \partial_\bullet\right) = H_\bullet^\infty(M; \mathbb{Z})$$

where $H_\bullet^\infty(M; \mathbb{Z})$ is the smooth singular homology defined in Appendix A.§2 of [22].

Since $H_\bullet^\infty(M; \mathbb{Z}) \cong H_\bullet(M; \mathbb{Z})$ by Theorem 2.1 of [22, Appendix A.§2],

$$H_\bullet\left(S_\bullet^\infty(M)/D_\bullet^\infty(M), \partial_\bullet\right) = H_\bullet(M; \mathbb{Z}).$$

2.2.4 Fibered Product of Topological Chains

Let P_1, P_2, B be topological spaces and let $\sigma_i : P_i \to B$ be continuous maps for $i = 1, 2$. The *fibered product* of σ_1 and σ_2 is defined as

$$P_1 \times_{\sigma_1,B,\sigma_2} P_2 := \{(x_1, x_2) \in P_1 \times P_2 \mid \sigma_1(x_1) = \sigma_2(x_2)\}$$

or, equivalently,

$$P_1 \times_{\sigma_1,B,\sigma_2} P_2 := (\sigma_1 \times \sigma_2)^{-1}(\Delta)$$

where $\Delta := \{(x, x) \mid x \in B\}$ is the diagonal of B. We will usually omit the maps σ_1, σ_2 from the notation and write $P_1 \times_B P_2 := P_1 \times_{\sigma_1,B,\sigma_2} P_2$.

Lemma 2.13 *Let $\sigma_i : P_i \to B$ be smooth maps for $i = 1, 2$ and P_i, B smooth manifolds of dimension p_i and b respectively. If σ_1 and σ_2 intersect transversely, then $P_1 \times_B P_2$ is a smooth manifold of dimension $p_1 + p_2 - b$.*

Proof Observe that σ_1 and σ_2 intersect transversely if and only if $\sigma_1 \times \sigma_2$ and Δ intersect transversely, which implies that $P_1 \times_B P_2 = (\sigma_1 \times \sigma_2)^{-1}(\Delta)$ is a smooth manifold. In addition

$$\dim P_1 \times_B P_2 = \dim\left((\sigma_1 \times \sigma_2) \pitchfork \Delta\right) = p_1 + p_2 - b. \qquad \square$$

Let $\{C_p\}_{p\geq 0}$ be a collection of topological spaces. We say that a topological space P has *degree* p if $P \in C_p$ (In our case, C_p will contain p-dimensional manifolds with corners). If B is a b-dimensional smooth manifold, $P_i \in C_{p_i}$ and $\sigma_i : P_i \to B$ we can associate the *degree* $p_1 + p_2 - b$ to $P_1 \times_B P_2$. In the case that P_1, P_2 are also smooth manifolds of dimension p_1 and p_2 respectively and the maps σ_1 intersect transversely σ_2, the degree of $P_1 \times_B P_2$ is the same as $\dim(P_1 \times_B P_2)$.

The collection $\{C_p\}_{p\geq 0}$ is said to be closed under fibered product with respect to some collection of maps if for every $P_1 \in C_{p_1}$ and $P_2 \in C_{p_2}$, then $P_1 \times_B P_2 \in C_{p_1+p_2-b}$.

We now define fibered products of abstract topological chains.

Definition 2.14 Assume that $\{C_p\}$ is closed under fibered product with respect to some collection of maps. Let $\sigma_i = \sum_k n_{i,k}\sigma_{i,k} \in S_{p_i}(B)$ for $i = 1, 2$ and $\sigma_{i,k} : P_{i,k} \to B$ are singular C_{p_i} spaces.

Define $P_i := \sum_k n_{i,k} P_{i,k} \in S_{p_i}$. Then the *fibered product* of σ_1 and σ_2 over B is defined as

$$P_1 \times_{\sigma_1,B,\sigma_2} P_2 := \sum_{k,j} n_{1,k} n_{2,j} P_{1,k} \times_{\sigma_{1,k},B,\sigma_{2,j}} P_{2,j} \in S_{p_1+p_2-b}.$$

To simplify the notation, we usually omit the maps σ_1, σ_2 from the notation and write $P_1 \times_B P_2$ instead of $P_1 \times_{\sigma_1,B,\sigma_2} P_2$. If $\sigma_1 = 0$ or $\sigma_2 = 0$, we define $P_1 \times_B P_2 = 0$.

The next lemma extends the boundary operator on $S_\bullet$ from Definition 2.8 to fibered products as well.

Definition 2.15 Let $\{C_p\}_{p\geq 0}$ be a collection of topological spaces and let $\{\tilde{C}_p\}_{p\geq 0}$ its closure under fibered product with respect to some collection of maps. Let $\tilde{S}_p$ be the free group generated by $\tilde{C}_p$. We extend the *boundary operator* $d : S_\bullet \to S_{\bullet-1}$ to

$$d : \tilde{S}_\bullet \to \tilde{S}_{\bullet-1}$$

by setting

$$d_{p_1+p_2-b}(P_1 \times_B P_2) = d(P_1) \times_B P_2 + (-1)^{p_1+b} P_1 \times_B d(P_2)$$

for $P_i \in S_{p_i}$ and $i = 1, 2$.

The boundary map on fibered products behaves well with multiple fibered products, as shown in the following lemma.

Lemma 2.16 *The boundary operator on fibered products in Definition 2.15 is a well defined boundary operator. That is, $d^2 : \tilde{S}_\bullet \to \tilde{S}_{\bullet-2}$ is the zero map.*

Moreover, $d : \tilde{S}_\bullet \to \tilde{S}_{\bullet-1}$ behaves well with multiple fibered products, that is,

$$d\left((P_1 \times_{B_1} P_2) \times_{B_2} P_3\right) = d\left(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)\right).$$

Proof The degree of both $d(P_1) \times_B P_2$ and $P_1 \times_B P_2$ is $p_1 + p_2 - b - 1$, so d reduces the degree by 1.

We compute $d^2(P_1 \times_B P_2)$:

$$\begin{aligned} d^2(P_1 \times_B P_2) &= d(d(P_1) \times_B P_2 + (-1)^{p_1+b} P_1 \times_B d(P_2)) \\ &= d^2(P_1) \times_B P_2 \\ &\quad + (-1)^{p_1+b}\big(- d(P_1) \times_B d(P_2) + d(P_1) \times_B d(P_2) \\ &\qquad\qquad + (-1)^{p_1+b} P_1 \times_B d^2(P_2)\big) \\ &= 0. \end{aligned}$$

Therefore, $d : \tilde{S}_\bullet \to \tilde{S}_{\bullet-1}$ is a well-defined boundary operator.

To show that

$$d\left((P_1 \times_{B_1} P_2) \times_{B_2} P_3\right) = d\left(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)\right),$$

we compute both sides:

$$\begin{aligned} d\left((P_1 \times_{B_1} P_2) \times_{B_2} P_3\right) &= d\left(P_1 \times_{B_1} P_2\right) \times_{B_2} P_3 \\ &\quad + (-1)^{p_1+p_2-b_1+b_2} P_1 \times_{B_1} P_2 \times_{B_2} dP_3 \\ &= d(P_1) \times_{B_1} P_2 \times_{B_2} P_3 + (-1)^{p_1+b_1} P_1 \times_{B_1} d(P_2) \times_{B_2} P_3 \end{aligned}$$

$$+ (-1)^{p_1+p_2-b_1+b_2} P_1 \times_{B_1} P_2 \times_{B_2} dP_3$$

$$\begin{aligned} d\left(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)\right) &= d(P_1) \times_{B_1} (P_2 \times_{B_2} P_3) \\ &\quad + (-1)^{p_1+b_1} P_1 \times_{B_1} d\left(P_2 \times_{B_2} P_3\right) \\ &= d(P_1) \times_{B_1} P_2 \times_{B_2} P_3 + (-1)^{p_1+b_1} P_1 \times_{B_1} d(P_2) \times_{B_2} P_3 \\ &\quad + (-1)^{p_1+p_2+b_1+b_2} P_1 \times_{B_1} P_2 \times_{B_2} dP_3 \end{aligned}$$

and since $(-1)^{b_1} = (-1)^{-b_1}$, we get that

$$d\left((P_1 \times_{B_1} P_2) \times_{B_2} P_3\right) = d\left(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)\right). \qquad \square$$

2.2.5 Compactificated Moduli Spaces

Let (M, g) be a compact Riemannian manifold and (f, X) a Morse-Bott-Smale pair. We denote by φ_t the flow of X. For every $B, B' \subset \mathrm{Crit}(f)$ critical submanifolds, the mapping $(t, x) \mapsto \varphi_t(x)$ induces a free $\mathbb{R}$-action on $W(B, B')$. Therefore, we can define

$$\mathcal{M}(B, B') := W(B, B')/\mathbb{R} = \left(W^u(B) \cap W^s(B')\right)/\mathbb{R}$$

to be the quotient space of flow lines from B to B'. We call $\mathcal{M}(B, B')$ the *moduli space of flow lines from B to B'.*

When we take a fibered product with $M(B, B')$, that is, a fibered product of the form

$$P \times_B \mathcal{M}(B, B') \text{ or } \mathcal{M}(B, B') \times_{B'} P,$$

we always take with respect to the beginning and endpoint maps

$$e_- : \mathcal{M}(B, B') \to B \quad \text{and} \quad e_+ : \mathcal{M}(B, B') \to B'.$$

Lemma 2.17 *The moduli space $\mathcal{M}(B, B')$ is a smooth manifold of dimension $\lambda_B - \lambda_{B'} - 1$.*

Proof $\mathcal{M}(B, B')$ is a quotient of a manifold by a free $\mathbb{R}$-action, and therefore a manifold. The dimension of $\mathcal{M}(B, B')$ is

$$\dim \mathcal{M}(B, B') = \dim W(B, B') - 1 = \lambda_B - \lambda_{B'} - 1. \qquad \square$$

Theorem 2.18 (Gluing) *[11, Appendix A.3, Theorem A.11] Let B, B', and B'' be critical submanifolds of f. Suppose that the following hold:*

1. *$W^u(B)$ and $W^s(B')$ intersect transversely.*
2. *$W^u(B')$ and $W^s(B'')$ intersect transversely.*

3. *For every $x \in B'$, $W^u(x)$ and $W^s(B'')$ intersect transversely.*

Then there is $\varepsilon > 0$ and an injective local diffeomorphism

$$G : \mathcal{M}(B, B') \times_{B'} \mathcal{M}(B', B'') \times (0, \varepsilon) \to \mathcal{M}(B, B'')$$

such that

$$\mathcal{M}(B, B'') \cong \mathcal{M}(B, B'') \setminus \mathrm{Im}(G).$$

That is, $\mathrm{Im}(G)$ *is "an end" of $\mathcal{M}(B, B'')$.*

An element $a \in \mathcal{M}(B_1, B_2) \times_{B_2} \cdots \times_{B_{n-1}} \mathcal{M}(B_{n-1}, B_n)$ for $n \geq 1$ is called a *piecewise gradient flow line from B_1 to B_2*.

Theorem 2.19 (Compactification) *[11, Lemma 3.3] Let $f : M \to \mathbb{R}$ be Morse-Bott-Smale and B, B' be critical submanifolds of f. Then $\mathcal{M}(B, B')$ has a compactification $\overline{\mathcal{M}}(B, B')$ consisting of all piecewise gradient flow lines from B to B'. More precisely,*

$$\overline{\mathcal{M}}(B, B') = \mathcal{M}(B, B') \cup \bigcup_{n \in \mathbb{N}} \bigcup_{B_1, \ldots, B_n} \mathcal{M}(B, B_1) \times_{B_1} \cdots \times_{B_n} \mathcal{M}(B_n, B')$$

where $B_1, \ldots, B_n$ are critical submanifolds of f and

$$\lambda_B > \lambda_{B_1} > \cdots > \lambda_{B_n} > \lambda_{B'}.$$

$\overline{\mathcal{M}}(B, B')$ is either empty or a smooth manifold with corners of dimension

$$\lambda_B - \lambda_{B'} + b - 1.$$

Moreover, the beginning and endpoint maps e_-, e_+ extend to smooth maps

$$e_- : \overline{\mathcal{M}}(B, B') \to B,\ e_+ : \overline{\mathcal{M}}(B, B') \to B'$$

where e_- has the structure of a locally trivial fiber bundle.

From now on, we define b_i as the dimension of a critical submanifold $B_i \subset \mathrm{Crit}(f)$ and i as its index. In addition, we denote by $\hat{B}_i$ the set of all critical points of index i. Equivalently,

$$\hat{B}_i = \bigcup_{B_i} B_i$$

where the union runs over all critical submanifolds of index i.

We define

$$\overline{\mathcal{M}}(B_i, B_k, B_{i-j}) := \overline{\mathcal{M}}(B_i, B_k) \times_{B_k} \overline{\mathcal{M}}(B_k, B_{i-j})$$

and similarly $\overline{\mathcal{M}}(B_i, B_s, B_k, B_{i-j})$ for $i - j < s < k < i$.

Since $W(B_i, B_{i-j})$ has dimension (and degree) $b_i + j$, the degree of $\mathcal{M}(B_i, B_{i-j})$ is defined to be $b_i + j - 1$ (which is also the dimension of $\mathcal{M}(B_i, B_{i-j})$). Therefore, the degree of $\overline{\mathcal{M}}(B_i, B_{i-j})$ can be defined to be $b_i + j - 1$ as well.

Definition 2.20 Let $f : M \to \mathbb{R}$ be a weakly self-indexing Morse-Bott function.

For every $p \geq 0$ we define C_p to be the set consisting of connected components of fibered products of the form

$$\overline{\mathcal{M}}(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \cdots \times_{B_{i_{n-1}}} \overline{\mathcal{M}}(B_{i_{n-1}, B_{i_n}})$$

with degree p, where $m \geq i_1 > \ldots > i_n \geq 0$. Let S_p be the free abelian group generated by C_p. Define $d : S_p \to S_{p-1}$ on generators by setting

$$d\overline{\mathcal{M}}(B_i, B_{i-j}) = (-1)^{i+b_i} \sum_{i-j<k<i} \sum_{B_k} \overline{\mathcal{M}}(B_i, B_k) \times_{B_k} \overline{\mathcal{M}}(B_k, B_{i-j})$$

(where the second sum runs over all critical submanifolds of index k).

Lemma 2.21 *The above-defined operator $d : S_p \to S_{p-1}$ is well-defined and it is a boundary operator, that is, $d \circ d = 0$.*

Proof First, observe that the degree of $\overline{\mathcal{M}}(B_i, B_{i-j})$ is $b_i + j - 1$, $\overline{\mathcal{M}}(B_i, B_k)$ has degree $i - k + b_i - 1$ (which does not depend on the dimension of B_k) and $\overline{\mathcal{M}}(B_k, B_{i-j})$ has degree $k - (i - j) + b_k - 1$. Therefore, by the definition of the degree of fibered products, $\overline{\mathcal{M}}(B_i, B_k, B_{i-j})$ has degree

$$(i - k + b_i - 1) + (k - (i - j) + b_k - 1) - b_k = b_i + j - 2$$

and therefore $d(\overline{\mathcal{M}}(B_i, B_{i-j}))$ is a linear sum of elements in S_{j+b_i-2}, so d is well-defined on $\overline{\mathcal{M}}(B_i, B_{i-j})$.
To show that $d^2 = 0$, we first calculate $d\overline{\mathcal{M}}(B_i, B_k, B_{i-j})$:

$$\begin{aligned} d\left(\overline{\mathcal{M}}(B_i, B_k, B_{i-j})\right) &= d\left(\overline{\mathcal{M}}(B_i, B_k)\right) \times_{B_k} \overline{\mathcal{M}}(B_k, B_{i-j}) \\ &\quad + (-1)^{i-k+b_i-1+k+b_k} \overline{\mathcal{M}}(B_i, B_k) \times_{B_k} d\left(\overline{\mathcal{M}}(B_k, B_{i-j})\right) \\ &= (-1)^{i+b_i} \sum_{k<s<i} \sum_{B_s} \overline{\mathcal{M}}(B_i, B_s, B_k, B_{i-j}) \\ &\quad + (-1)^{i+b_i-1} \sum_{i-j<s<k} \sum_{B_s} \overline{\mathcal{M}}(B_i, B_k, B_s, B_{i-j}) \end{aligned}$$

and hence

$$\begin{aligned} d^2\left(\overline{\mathcal{M}}(B_i, B_{i-j})\right) &= (-1)^{i+b_i} \sum_{i-j<k<i} \sum_{B_k} d\left(\overline{\mathcal{M}}(B_i, B_k, B_{i-j})\right) \\ &= \sum_{i-j<k<i} \sum_{B_k} \Bigg[\sum_{k<s<i} \sum_{B_s} \overline{\mathcal{M}}(B_i, B_s, B_k, B_{i-j}) \\ &\qquad - \sum_{i-j<s<k} \sum_{B_s} \overline{\mathcal{M}}(B_i, B_k, B_s, B_{i-j}) \Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i-j<k<i} \sum_{B_k} \sum_{i-j<s<k} \sum_{B_s} \Big[\overline{\mathcal{M}}(B_i, B_s, B_k, B_{i-j}) \\
&\qquad\qquad\qquad\qquad\qquad - \overline{\mathcal{M}}(B_i, B_s, B_k, B_{i-j}) \Big] \\
&= 0.
\end{aligned}
$$

Therefore, d is a well-defined boundary operator. □

Union Of Fibered products

Let $P_1, P_2, P_1', P_2', B, B'$ be pairwise disjoint topological spaces, and

$$\sigma_i : P_i \to B, \, \sigma_i' : P_i' \to B'$$

be continuous maps for $i = 1, 2$. Then

$$
\begin{aligned}
(P_1 \cup P_1') \times_{B \cup B'} (P_2 \cup P_2') &= \left(\left((\sigma_1 \cup \sigma_1') \times (\sigma_2 \cup \sigma_2') \right)^{-1} (\Delta(B \cup B')) \right) \\
&= (\sigma_1 \times \sigma_2)^{-1}(\Delta(B)) \cup (\sigma_1' \times \sigma_2')^{-1}(\Delta(B')) \\
&= (P_1 \times_B P_2) \cup (P_1' \times_{B'} P_2')
\end{aligned}
$$

and the union is disjoint. Therefore, since $\hat{B}$ is a finite union of critical submanifolds, we can define

$$\mathcal{M}(B_i, \hat{B}_j) = \bigcup_{B_j} \mathcal{M}(B, B_j)$$

(where the union runs over all critical submanifolds of index j) and analogously for $\mathcal{M}(\hat{B}_i, B_j)$ and $\mathcal{M}(\hat{B}_i, \hat{B}_j)$. The definition of the compactificated moduli space $\overline{\mathcal{M}}(B_i, B_j)$ can be extended analoguously as well.

In terms of topological chains, if C_p is the set of all elements of the form $\mathcal{M}(B_i, B_j)$ with degree p, we can associate $\overline{\mathcal{M}}(B_i, \hat{B}_k, B_{i-j})$ with the abstract topological chain $\sum_{B_k} \overline{\mathcal{M}}(B, B_k, B_{i-j})$, which all have the same degree by the proof of the above lemma.

2.3 Morse-Bott-Smale Chain Complex

In this section, we define the Morse-Bott-Smale chain complex. The chain complex consists of smooth maps from faces of I^N and fibered products of faces with compactificated moduli spaces. Next, we define degeneracy conditions on the chain complex. Then, we define the Morse-Bott homology of a Morse-Bott function $f : M \to \mathbb{R}$. In the next section, we show that the homology of the chain complex is invariant of the function. This section follows section 5 of [1].

In this section we assume that M is compact and oriented, and $f : M \to \mathbb{R}$ is a Morse-Bott-Smale function with respect to some gradient-like vector field X. In addition, we assume that all critical submanifolds B and their negative normal bundle $\nu_*^-(B)$ are oriented.

2.3.1 Manifolds With Corners

A *topological manifold with corners* is a second countable Hausdorff topological space V so that every $x \in V$ has a neighborhood $x \in U_x \subset V$ and a homeomorphism $\varphi_x : U_x \to \mathbb{R}^k \times [0,\infty)^{m-k} =: \mathbb{R}^m_k$. Such a pair (U_x, φ_x) is called a *chart*. We say that two charts (U_x, φ_x), (U_y, φ_y) are *(C^∞)-compatible* if the transition maps

$$\varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \to \varphi_y(U_x \cap U_y)$$

and

$$\varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \to \varphi_y(U_x \cap U_y)$$

are both C^∞.

Definition 2.22 A (smooth) manifold with corners is a topological manifold with corners V such that for every $x, y \in V$ the charts (U_x, φ_x) and (U_y, φ_y) are C^∞-compatible.

Let V be a manifold with corners and let (U_x, φ_x) be a chart around $x \in V$ so that $\varphi_x(x) = 0 \in \mathbb{R}^m_k$. Then k does not depend on the choice of (U_x, φ_x) (since $\mathbb{R}^m_k$ and $\mathbb{R}^m_l$ are not diffeomorphic if $k \neq l$). Therefore, we can define the *index* of x in V to be k and denote if by $\mathrm{Ind}(x, V)$. For $k = 0, \ldots, m$, we define V_k to be the submanifold of index k of V. i.e.

$$V_k := \{x \in V \mid \mathrm{Ind}(x, V) = k\}.$$

Note that V_k is a k-dimensional submanifold (without boundary) of V.

Definition 2.23 A *k-stratum* of V is a connected component of V_k. We define V_n to be the *interior* of V, and $V \setminus V_n$ to be the *boundary* of V.

Recall (Lemma 2.17) that $\mathcal{M}(B_i, B_{i-j})$ is an $(b_i + j - 1)$-dimensional manifold. Using Lemma 2.13, we get that

$$\mathcal{M}(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \mathcal{M}(B_{i_2}, B_{i_3}) \times_{B_{i_3}} \cdots \times_{B_{i_n}} \mathcal{M}(B_{i_n}, B_{i_{n+1}})$$

is a smooth manifold of dimension

$$\sum_{j=1}^{n}(b_{i_j} + i_j - i_{j+1} - 1) - \sum_{j=2}^{n} b_{i_j} = b_{i_1} + i_1 - i_n - n.$$

Therefore, $\overline{\mathcal{M}}(B_i, B_{i-j})_k$ consists of all fibered products

$$\mathcal{M}(B_i, B_{i_2}) \times_{B_{i_2}} \mathcal{M}(B_{k_2}, B_{i_3}) \times_{B_{i_3}} \cdots \times_{B_{i_n}} \mathcal{M}(B_{i_n}, B_{i-j})$$

where $b_i + j - n = k$, or $n = b_i + j - k$. That is, $\overline{\mathcal{M}}(B_i, B_{i-j})_k$ is the submanifold consists of all piecewise gradient flow lines passing through $b_i + j - k$ intermediate critical submanifolds.

Given an orientation on V_m, the interior of V, we can extend the orientation to V.

Lemma 2.24 *Let V be an oriented smooth manifold with corners. Then the orientation on V_m defines and orientation on V.*

Proof By the collar theorem for manifold with corners [23, Lemma 2.1.6], there is an embedding $i : \partial V \times [0, 1) \hookrightarrow V$. Now, let $\xi : [0, 1] \to [1/2, 1]$ be a diffeomorphism such that $\xi(t) = t$ if $t \geq 3/4$. We define

$$W := V \setminus i(\partial V \times [0, 1/2))$$

and $\psi : V \to W$ by

$$\psi(x, t) = (x, \xi(t))$$

for $(x, t) \in \partial V \times [0, 1)$ and $\psi(x) = x$ if $x \notin V \times [0, 1)$. ψ is a smooth bijection, and also a local diffeomorphism. Hence ψ is a diffeomorphism between V and W. Now, let $O := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a collection of orientation-preserving charts on V_m that covers V_m (such O exists because V_m is orientable). Hence, O defines an orientation on V_m. Then

$$O' = \{(U_\alpha \cap W, \varphi_\alpha|_W)\}_{\alpha \in A}$$

is a collection of orientation-preserving charts on W that covers W, and hence induces an orientation on W. Since ψ is an orientation-preserving diffeomorphism (because it is the identity on an open set), we get an orientation on V. $\square$

For $x \in V$, we define $\hat{T}_x V$ to be the tangent space at x of the stratum containing x.

Definition 2.25 Let V, W be smooth manifolds with corners and let $A \subset Y$ be a submanifold with corners. We say that $f : V \to W$ intersects A *stratum transversely* if

$$df_x(\hat{T}_x V) \oplus \hat{T}_{f(x)} A = \hat{T}_{f(x)} W$$

for every $x \in f^{-1}(A)$.

We will use the following theorem:

Theorem 2.26 *[24, Theorem 3] Let V be a manifold with corners and W be a manifold without boundary. Let $A \subset W$ be a submanifold with corners and $f : V \to W$ be a smooth map that intersects A transversely and stratum transversely. Assume that $f^{-1}(A) \neq \emptyset$. Then:*

1. *$f^{-1}(A) \subset V$ is a smooth submanifold with corners.*
2. $\dim X - \dim f^{-1}(A) = \dim Y - \dim A$.
3. *For every $x \in f^{-1}(A)$,*

$$\operatorname{Ind}(x, V) - \operatorname{Ind}(x, f^{-1}(A)) = \operatorname{Ind}(f(x), W) - \operatorname{Ind}(f(x), A).$$

2.3.2 Complex

We set C_p to be the set consisting of p-faces of I^N and the connected components of fibered products with dimension p of the form

$$Q' = Q \times_{\sigma,\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_{i_1}, \hat{B}_{i_2}) \times_{\hat{B}_{i_2}} \cdots \times_{\hat{B}_{i_{n-1}}} \overline{\mathcal{M}}(\hat{B}_{i_{n-1}}, \hat{B}_{i_n})$$

where $i_1 > i_2 > \dots i_n \geq 0$, Q is a face of dimension $q \leq p$, and $\sigma : Q \to B_{i_1}$ are smooth. The fibered products are taken with respect to e_- and e_+ (for $\overline{\mathcal{M}}(\hat{B}_{i_j}, \hat{B}_{i_{j+1}})$). Observe that Q' has degree $q + i_1 - i_n - n$.

Lemma 2.27 *[1, Lemma 5.1] The objects in C_p are compact oriented manifolds with corners.*

Definition 2.28 Let S_p be the free abelian group generated by C_p. We define $S_p^\infty(\hat{B}_i)$ to be the subgroup of $S_p(\hat{B}_i)$ generated by the maps $(\sigma_P : P \to \hat{B}_i)$ for some $P \in C_p$ satisfying the following conditions:

1. σ_P is smooth.
2. if P is a connected component of a fibered product, then $\sigma_P = e_+ \circ \pi$, where

$$\pi : Q \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_{i_1}, \hat{B}_{i_2}) \times_{\hat{B}_{i_2}} \cdots \times_{\hat{B}_{i_{n-1}}} \overline{\mathcal{M}}(\hat{B}_{i_{n-1}}, \hat{B}_{i_n}) \to \overline{\mathcal{M}}(\hat{B}_{i_{n-1}}, \hat{B}_{i_n})$$

   is the projection to the last component of the fibered product.

   $e_+ \circ \pi$ can be described geometrically as the endpoint map of a piecewise gradient line.

We will later (Definition 2.37) quotient $S_p^\infty(\hat{B}_i)$ by a subgroup of $D_p^\infty(\hat{B}_i) \subset S_p^\infty(\hat{B}_i)$, creating an identification between maps from connected components of fibered products to $\sum_\alpha n_\alpha \sigma_\alpha \in S_p^\infty(\hat{B}_i)$ where each $\sigma_\alpha : P_\alpha \to \hat{B}_i$ is a smooth map from p-faces of I^N.

We say that the elements of $S_p^\infty(\hat{B}_i)$ have *Morse-Bott degree $p + i$*.

Definition 2.29 For every $0 \leq k \leq m$, we define $\tilde{C}_k$ to be free group generated by all smooth singular topological chains of Morse-Bott degree k. i.e.

$$\tilde{C}_k(f) := \bigoplus_{i=0}^{k} S^{\infty}_{k-i}(\hat{B}_i).$$

Let P be a topological space and let $\sigma_P : P \to \hat{B}_i$ be a smooth map. Every point in the fibered product

$$P \times_{\hat{B}^i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j})$$

can be described as a pair (x, y), where $x \in P$ and y is a piecewise gradient flow line from $\sigma_P(x)$ to $\hat{B}_{i-j}$. Hence, the map

$$\sigma_j : P \times_{\hat{B}^i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \to \hat{B}_j$$

defined by $\sigma_j = \pi_2 \circ e_+$ sends such (x, y) to $e_+(y)$.

Lemma 2.30 *[1, Lemma 5.3] Let $\sigma_P : P \to \hat{B}_i$ be a singular C_p-space in $S^{\infty}_p(\hat{B}_i)$ and let $1 \leq j \leq i$. Then we can identify $P \times_{\sigma_P, \hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j})$ with*

$$\sum_k n_k R_k \in S_{p+j-1}$$

where R_k is a connected component of $P \times_{\sigma_P, \hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j})$, and n_k is the sign induced from the orientation if $\dim R_k = 0$ and 1 otherwise.

Proof $\hat{B}_i$ is a union of submanifolds of M. $\sigma_P : P \to \hat{B}_i$ is continuous, P is connected and so $\sigma_P(P)$ lies in a connected component B of $\hat{B}_i$. Now, $\overline{\mathcal{M}}(B, \hat{B}_{i-j})$ is an abstract topological chain of degree $j + b - 1$, so

$$P \times_{\sigma_P, B} \overline{\mathcal{M}}(B, \hat{B}_{i-j})$$

has degree $p + j - 1$ (which is independent of the degree of B).

Let B' be a connected component of $\hat{B}_{i-j}$. Since $P \in C_p$ is a compact manifold with corners, $\overline{\mathcal{M}}(B, B')$ is also a compact manifold with corners. By 2.27 $P \times_B \overline{\mathcal{M}}(B, B')$ is also a compact manifold with corners, so it has finitely many components. Thus $P \times_B \overline{\mathcal{M}}(B, \hat{B}_{i_j})$ has finitely many components. $\square$

Using the above lemma, we can define a homomorphism

$$\partial_j : S^{\infty}_p(\hat{B}_i) \to S^{\infty}_{p+j-1}(B_{i-j}).$$

We define ∂_j on a generator $\sigma_P : P \to \hat{B}_i$ by $\partial_j(\sigma_P) = \sigma_R$, where $\sigma_R = \sum_k n_k \sigma_{R_k}$ and extend linearly. Note that since R_k is a connected component of a fibered product, there is only one map

$$(\sigma : R_k \to \hat{B}_{i-j}) \in S^{\infty}_{p+j-1}(B_j),$$

which is $\sigma_{R_k} = e_+ \circ \pi_2$. ∂_j decreases the Morse-Bott degree by 1. We define $\partial_j : S_p^\infty(\hat{B}_i) \to S_{p+j-1}^\infty(B_{i-j})$ to be the zero map if $j > i$.

We can now define $\partial : \tilde{C}_k(f) \to \tilde{C}_{k-1}(f)$ by

$$\partial(\sigma) = \bigoplus_{i=0}^{m} \partial_j(\sigma)$$

where $\sigma \in S_{k-1}^\infty(B_i)$, $\partial_0 = (-1)^k \cdot \partial$ (where ∂ is the boundary operator defined in Section 2.2) and ∂_j is defined as above for $1 \leq j \leq i$.

Proposition 2.31 *For $0 \leq j \leq m$, $\sum_{q=0}^{j-q} \partial_q \partial_{j-q} = 0$. This means that $(\tilde{C}_\bullet(f), \partial)$ is a chain complex.*

Proof The case $j = 0$ is the singular homology. Let $\sigma_P \in S_p^\infty(\hat{B}_i)$ be a singular C_p-space of $\hat{B}_i$. Since

$$\partial_q \circ \partial_{j-q} : S_p^\infty(\hat{B}_i) \to S_{p+j-2}^\infty(\hat{B}_{i-j})$$

is the zero map if $j > i$, assume $1 \leq j \leq i$.

We are going to compute each component of $\sum_{j=1}^{i} \partial_q(\partial_{j-q}(\sigma_P))$ separately. We are going to write the computations in terms of abstract topological chains. For a singular C_p-space $\sigma : P \to \hat{B}_i$, we write

$$d_j(P) = R \in S_{p+j-1}$$

if $\partial_j(\sigma_P) = \sigma_R$. This is allowed because only one map is used from every domain in the computations below.

We divide into 3 cases: $q = 0$, $1 \leq q < j$, and $q = j$. If $q = 0$, then

$$\begin{aligned} d_0(d_j(P)) &= d_0\left(P \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j})\right) = d_0\left(\sum_{B_i} P \times_{B_i} \overline{\mathcal{M}}(B_i, \hat{B}_{i-j})\right) \\ &= (-1)^{p+i-1}\Bigg(d(P) \times_{B_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \\ &\qquad + \sum_{B_i} (-1)^{p+b_i} P \times_{B_i} d\overline{\mathcal{M}}(B_i, \hat{B}_{i-j})\Bigg) \\ &= (-1)^{p+i-1}\Bigg(d(P) \times_{B_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \\ &\qquad + \sum_{B_i} (-1)^{p+b_i+b_i+i} \sum_{q=1}^{j-1} P \times_{B_i} \overline{\mathcal{M}}(B_i, \hat{B}_{i-q}, \hat{B}_{i-j})\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{p+i-1}\Big(d(P) \times_{B_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \\
&\qquad\qquad + (-1)^{p+i} \sum_{q=1}^{j-1} \sum_{B_i} P \times_{B_i} \overline{\mathcal{M}}(B_i, \hat{B}_{i-q}, \hat{B}_{i-j})\Big) \\
&= (-1)^{p+i-1} d(P) \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) - \sum_{q=1}^{j-1} P \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-q}, \hat{B}_{i-j}).
\end{aligned}
$$

If $1 \le q \le j-1$, then

$$
d_q\left(d_{j-q}(P)\right) = P \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j+q}, \hat{B}_{i-j}).
$$

If $q = j$, then

$$
d_j(d_0(P)) = d_j\left((-1)^{p+i} d(P)\right) = (-1)^{p+i}(dP \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j})).
$$

Summing everything yields

$$
\begin{aligned}
\sum_{q=0}^{j} d_q\left(d_{j-q}(P)\right) &= (-1)^{p+i-1} d(P) \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j}) \\
&\quad - \sum_{q=1}^{j-1} P \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_q, \hat{B}_{i-j}) \\
&\quad + \sum_{q=1}^{j-1} P \times_{\hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_{i-j+q}, \hat{B}_{i-j}) \\
&\quad + (-1)^{p+i}(d(P) \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})) \\
&= 0
\end{aligned}
$$

□

2.3.3 Orientations

Let $B \subset \mathrm{Crit}(f)$ be a critical submanifold. Recall that we assume that $\nu_*^-(B)$ and B are oriented. The relation

$$
T_p M = T_p B \oplus \nu_p^-(B) \oplus \nu_p^+(B)
$$

determines an orientation on $\nu_p^+(B)$. The maps $E^- : \nu_*^-(B) \to W^u(B)$ and $E^+ : \nu_*^+(B) \to W^s(B)$ (from Theorem 2.5) are bijective immersions, and hence induce orientation on $W^u(B)$ and $W^s(B)$, respectively.

For B, B' connected critical submanifolds, the orientation on $W(B, B')$ is determined by

$$
T_x M = T_x W(B, B') \oplus \nu_x\left(W^s(B')\right) \oplus \nu_x(W^u(B))
$$

for $x \in W(B, B')$.

For a regular value $f(B') < a < f(B)$ we can identify $\mathcal{M}(B, B') = f^{-1}(a) \cap W(B, B')$ and get an orientation on $\mathcal{M}(B, B')$ using

$$T_x W(B, B') = \operatorname{span}\left((-\nabla f)(x)\right) \oplus T_x \mathcal{M}(B, B')$$

for all $x \in f^{-1}(a) \cap W(B, B')$.

The orientation on $\mathcal{M}(B, B')$ can be extended to $\overline{\mathcal{M}}(B, B')$ using Lemma 2.24.

Definition 2.32 Let P_1, P_2 be smooth manifolds with corners, and B be an oriented manifold without boundary. Assume that $\sigma_i : P_i \to B$ intersect transversely and stratum transversely. We define an orientation on $P_1 \times_B P_2$ by

$$(-1)^{b \cdot p_2} T_*(P_1 \times_B P_2) \oplus (\sigma_1 \times \sigma_2)^* \left(\nu_* \left(\Delta(B)\right)\right) = T_*(P_1 \times P_2)$$

where $\Delta(B) = \{(x, x) \mid x \in B\}$ and $\nu_*(\Delta(B))$ denotes the normal bundle of $\Delta(B)$ in $B \times B$.

Lemma 2.33 *[1, Lemma 5.8] The orientation defined above is associative. That is, the orientations induced on $(P_1 \times_{B_1} P_2) \times_{B_2} P_3$ and $P_1 \times_{B_1} (P_2 \times_{B_2} P_3)$ are the same.*

Proof Let $\sigma_1 : P_1 \to B_1$, $\sigma_2 : P_2 \to B_1$, $\sigma_2' : P_2 \to B_2$, and $\sigma_3 : P_3 \to B_2$ be smooth maps. We take the fibered products with respect to these maps.

We start by computing the orientation of $(P_1 \times_{B_1} P_2) \times_{B_2} P_3$. By Definition 2.32,

$$\begin{aligned}(-1)^{b_2 p_3} T_*((P_1 \times_{B_1} P_2) \times_{B_2} P_3) \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) \\ = T_*((P_1 \times_{B_1} P_2) \times P_3) = T_*((P_1 \times_{B_1} P_2)) \oplus T_*(P_3).\end{aligned}$$

Applying the definition again for $P_1 \times_{B_1} P_2$ gives

$$(-1)^{b_1 \cdot p_2} T_*(P_1 \times_{B_1} P_2) \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))) = T_*(P_1 \times P_2).$$

Therefore,

$$\begin{aligned} T_*(P_1 \times P_2) \oplus T_*(P_3) &= (-1)^{b_1 \cdot p_2} T_*(P_1 \times_{B_1} P_2) \\ &\qquad \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))) \oplus T_*(P_3) \\ &= (-1)^{b_1 \cdot p_2 + b_1 p_3} T_*(P_1 \times_{B_1} P_2) \\ &\qquad \oplus T_*(P_3) \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))). \end{aligned}$$

Putting these equations together yields:

$$
\begin{aligned}
(-1)^{b_2 p_3 + b_1 p_2 + b_1 p_3} & T_*((P_1 \times_{B_1} P_2) \times_{B_2} P_3) \\
& \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) \\
& \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))) = T_*((P_1 \times P_2) \times P_3) \\
& \qquad\qquad\qquad\qquad\qquad\quad = T_*(P_1) \oplus T_*(P_2) \oplus T_*(P_3).
\end{aligned}
$$

Now, we compute the orientation of $P_1 \times_{B_1} (P_2 \times_{B_2} P_3)$:

$$
\begin{aligned}
(-1)^{b_1(p_2+p_3-b_2)} & T_*(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)) \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))) \\
& = T_*(P_1 \times (P_2 \times_{B_2} P_3)) = T_*(P_1) \oplus T_*(P_2 \times_{B_2} P_3).
\end{aligned}
$$

Applying the definition again for $P_1 \times_{B_1} P_2$ gives

$$
(-1)^{b_2 \cdot p_3} T_*(P_2 \times_{B_2} P_3) \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) = T_*(P_2 \times P_3).
$$

and combining yields

$$
\begin{aligned}
(-1)^{b_1(p_2+p_3-b_2)+b_2 p_3} & T_*(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)) \\
& \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))) \\
& \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) = T_*(P_1 \times (P_2 \times P_3)) \\
& \qquad\qquad\qquad\qquad\qquad\quad = T_*(P_1) \oplus T_*(P_2) \oplus T_*(P_3).
\end{aligned}
$$

The coefficient of the orientation is

$$
(-1)^{b_1(p_2+p_3-b_2)+b_2 p_3} = (-1)^{b_1 p_2 + b_1 p_3 - b_1 p_2 + b_2 p_3}.
$$

Now, if we want to exchange places between $(\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1)))$ and $(\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2)))$, we have to multiply by $(-1)^{b_1 b_2}$. Therefore,

$$
\begin{aligned}
& (-1)^{b_1 p_2 + b_1 p_3 - b_1 p_2 + b_2 p_3} T_*(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)) \\
& \qquad \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))) \\
& \qquad \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) \\
& = (-1)^{b_1 p_2 + b_1 p_3 + b_2 p_3} T_*(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)) \\
& \qquad \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) \\
& \qquad \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1)))
\end{aligned}
$$

Together, we got that

$$
\begin{aligned}
& (-1)^{b_2 p_3 + b_1 p_2 + b_1 p_3} T_*((P_1 \times_{B_1} P_2) \times_{B_2} P_3) \\
& \qquad\qquad \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))) \\
= & (-1)^{b_1 p_2 + b_1 p_3 + b_2 p_3} T_*(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)) \\
& \qquad\qquad \oplus (\sigma_2' \times \sigma_3)^*(\nu_*(\Delta(B_2))) \oplus (\sigma_1 \times \sigma_2)^*(\nu_*(\Delta(B_1))).
\end{aligned}
$$

Therefore,

$$
T_*((P_1 \times_{B_1} P_2) \times_{B_2} P_3) = T_*(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)). \qquad \square
$$

Using the above lemma, we get a well-defined orientation on the connected components of

$$Q \times_{\hat{B}_{i_1}} \overline{\mathcal{M}}(\hat{B}_{i_1}, \hat{B}_{i_2}) \times_{\hat{B}_{i_2}} \cdots \times_{\hat{B}_{i_{n-1}}} \overline{\mathcal{M}}(\hat{B}_{i_{n-1}}, \hat{B}_{i_n})$$

where Q is a face of I^N.

2.3.4 Degenerate Singular Topological Chains

Definition 2.34 Let $\sigma_P, \sigma_Q \in S_p^\infty(\hat{B}_i)$ be singular C_p-spaces in $\hat{B}_i$ and let $d(Q) = \sum_j n_j Q_j \in S_{p-1}$. For a continuous map

$$\alpha : P \to Q,$$

let $\partial_0 \sigma_Q \circ \alpha$ denote the formal sum

$$(-1)^{p+i} \sum_j n_j \ (\sigma_Q \circ \alpha)|_{\alpha^{-1}(Q_j)} .$$

Define $D_p^\infty(B_i) \subset S_p^\infty(B_i)$, the set of *degenerate singular topological chains*, by the following conditions, called *degeneracy conditions*:

1. If α is an orientation preserving homeomorphism such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_0 \sigma_Q \circ \alpha = \partial_0 \sigma_P$, then $\sigma_Q - \sigma_P \in D_p^\infty(\hat{B}_i)$.

2. If P is a face of I^N and σ_P does not depend on a free coordinate of P, then $\sigma_P \in D_p^\infty(\hat{B}_i)$ and $\partial_j(\sigma_P) \in D_{p+j-1}^\infty(\hat{B}_{i-j})$ for every $j = 1, \ldots, m$.

3. If P and Q are connected components of fibered products and α is an orientation reversing homeomorphism such that $\sigma_Q \circ \alpha = \sigma_P$ and $\partial_0 \sigma_Q \circ \alpha = \partial_0 \sigma_P$, then $\sigma_Q + \sigma_P \in D_p^\infty(\hat{B}_i)$.

4. If Q is a face of I^N and R is a connected component of

$$Q \times_{\hat{B}_{i_1}} \overline{\mathcal{M}}(\hat{B}_{i_1}, \hat{B}_{i_2}) \times_{\hat{B}_{i_2}} \cdots \times_{\hat{B}_{i_{n-1}}} \overline{\mathcal{M}}(\hat{B}_{i_{n-1}}, \hat{B}_{i_n})$$

   with $\deg R > \dim B_{i_n}$ for every connected component B_{i_n} of $\hat{B}_{i_n}$, then $\sigma_R \in D_r^\infty(\hat{B}_{i_n})$ and $\partial_j(\sigma_R) \in D_{r+j-1}^\infty(\hat{B}_{i_n - j})$ for every $j = 0, \ldots, m$.

5. If $\sum_\alpha n_\alpha \sigma_\alpha \in S_*(R)$ is a smooth singular chain in a connected component R of a fibered product that represents the fundamental class of R and $(-1)^{r-i_n} \partial_0 \sigma_R - \sum_\alpha n_\alpha \partial(\sigma_R \circ \sigma_\alpha)$ is in $D_{r-1}^\infty(\hat{B}_i)$ as per conditions (1)-(5), then

$$\sigma_{R'} := \sigma_R - \sum_\alpha n_\alpha (\sigma_R \circ \sigma_\alpha) \in D_r^\infty(B_{i_n})$$

   and $\partial_j(\sigma_{R'}) \in D_{r+j-1}(B_{i_n - j})$ for every $j = 1, \ldots, m$.

The reasoning behind the degeneracy conditions is as follows: conditions (1) and (2) are similar to the degeneracy conditions in Definition 2.10, so that the chain complex $(S_\bullet^\infty(\hat{B}_i)/D_\bullet^\infty(\hat{B}_i), \partial_0)$ computes the singular homology of $\hat{B}_i$, by Theorem 2.11. In addition, conditions (1) and (3) define an identification between connected components of fibered products, based on their orientation. Condition (4) is a dimension limit, and condition (5) provides a way to identify between singular chains from fibered products and from faces of I^N.

Lemma 2.35 *[1, Lemma 5.10] For every $i, j = 0, \ldots, m$, the homomorphism*

$$\partial_j : S_p^\infty(\hat{B}_i) \to S_{p+j-1}^\infty(\hat{B}_{i-j})$$

induces a homomorphism

$$\partial_j : S_p^\infty(\hat{B}_i)/D_p^\infty(\hat{B}_i) \to S_{p+j-1}^\infty(\hat{B}_{i-j})/D_{p+j-1}^\infty(\hat{B}_{i-j}).$$

The following lemma shows that for every $\sigma_R : R \to \hat{B}_i$ where R is a connected component of a fibered product, there is an equivalent smooth singular chain $\sigma = \sum_\beta n_\beta \sigma_\beta$ where $\sigma_\beta : P_\beta \to \hat{B}_i$ are smooth singular chains and P_β are faces of I^N.

Lemma 2.36 *Let $R \in C_r$ be a connected component of a fibered product*

$$P \times_{\hat{B}_{i_1}} \overline{\mathcal{M}}(\hat{B}_{i_1}, \hat{B}_{i_2}) \times_{\hat{B}_{i_2}} \cdots \times_{\hat{B}_{i_{n-1}}} \overline{\mathcal{M}}(\hat{B}_{i_{n-1}}, \hat{B}_{i_n})$$

with $\deg R = r$. Then there is a smooth singular chain $\sigma = \sum_\alpha n_\alpha \sigma_\alpha$ such that $\sigma_R - \sigma \in D_r^\infty(\hat{B}_{i_n})$ and:

1. *$\sigma_\alpha : P_\alpha \to \hat{B}_{i_n}$ is a smooth singular chain and P_α is an r-face of I^N.*
2. *$\sigma_\alpha = \sigma_R \circ \tilde{\sigma}_\alpha$ where $\tilde{\sigma}_\alpha : P_\alpha \to R$ is a smooth singular chain in R for all α.*
3. *$\sum_\alpha n_\alpha \tilde{\sigma}_\alpha$ represents the fundamental class of R. That is, $\sum_\alpha n_\alpha \tilde{\sigma}_\alpha$ is a relative cycle in $S_p(R, \partial R) = S_r(R)/S_r(\partial R)$ which is a generator of $H_r(R, \partial R)$.*

Proof We prove by induction on r. If $r = 0$, then R is a connected 0-dimensional manifold with corners, and hence a point.

Therefore, $\partial_0 \sigma_R = 0$, and $\sigma : \{0\} \to R$ represents of the fundamental class of R. Also $\partial(\sigma_R \circ \sigma) = 0$, so by the fifth condition of degeneracy in Definition 2.34,

$$\sigma_R - \sigma_R \circ \sigma \in D_0^\infty(\hat{B}_{i_n}).$$

Assume now that $r \geq 1$. Write $\partial_R = \sum_k n_k R_k$ where $R_k \in C_{r-1}$. Using the induction hypothesis, for every R_k there is a singular chain

$$\sigma_k = \sum_{j_k} n_{j_k} \sigma_{j_k} \in S_{r-1}(R_k)$$

that satisfies the conditions of the lemma. Then

$$\begin{aligned}\partial_0\sigma_R - (-1)^{r+i_n}\sum_k n_k\sigma_k &= (-1)^{r+i_n}\left(\sum_k n_k\sigma_{R_k} - \sum_k n_k\sum_{j_k} n_{j_k}\sigma_{j_k}\right)\\ &= (-1)^{r+i_n}\left(\sum_k n_k\left(\sigma_{R_k} - \sum_{j_k} n_{j_k}\sigma_{j_k}\right)\right) \in D^\infty_{r-1}(B_{i_n}).\end{aligned}$$

By the induction hypothesis, $\sum_{j_k} n_{j_k}\tilde{\sigma}_{j_k}$ is a relative cycle in $S_{r-1}(R_k, \partial R_k)$ that represents the fundamental class of R_k. Therefore,

$$\sum_k n_k \sum_{j_k} n_{j_k}\tilde{\sigma}_{j_k} \in S_{r-1}(\partial R)$$

is a relative cycle in $S_{r-1}(\partial R, \partial(\partial R)) = S_{r-1}(\partial R)$ that represents the fundamental class of ∂R. By Lemma VI.9.1 of [25], there is a singular chain $\sum_\alpha n_\alpha\sigma_\alpha \in S_r(R)$ (which can be made smooth by perturbing the interior) such that

$$\partial\left(\sum_\alpha n_\alpha\sigma_\alpha\right) = \sum_k n_k\sigma_k \in S_r(R).$$

Since

$$\partial_0\sigma_R - (-1)^{r+i_n}\sum_k n_k\sigma_k \in D^\infty_{r-1}(B_{i_n}),$$

we get that

$$\sigma_R - \sum_\alpha n_\alpha(\sigma_R\circ\sigma_\alpha) \in D_r(B_{i_n}).$$

□

2.3.5 Homology

In this subsection, we define the Morse-Bott homology. Let $f : M \to \mathbb{R}$ be a Morse-Bott-Smale function. Recall that we defined C_p in Subsection 2.3.2 to be the set consisting of p-faces of I^N, as well as connected components of fibered products with degree p of the form

$$Q \times_{\sigma_Q, \hat{B}_{i_1}} \overline{\mathcal{M}}(\hat{B}_{i_1}, \hat{B}_{i_2}) \times_{\hat{B}_{i_2}} \cdots \times_{\hat{B}_{i_{n-1}}} \overline{\mathcal{M}}(\hat{B}_{i_{n-1}}, \hat{B}_{i_n})$$

where $\sigma_Q : Q \to \hat{B}_{i_1}$ is a smooth map from a face of dimension $q \geq p$ and $m \geq i_1 > \cdots > i_n \geq 0$.

Lemma 2.27 states that all elements in C_p are compact manifolds with corners. We defined S_p to be the free abelian group generated C_p. We defined $S^\infty_p(\hat{B}_i)$ in Definition 2.28 to be the subgroup generated by smmoth maps $\sigma_P :\to \hat{B}_i$ where P is a p-face of I^N.

However, if P is a connected component of a fibered product, the only map $\sigma_P : P \to \hat{B}_i$ in $S_p^\infty(\hat{B}_i)$ is $\sigma_P = e_+ \circ \pi$, where π is the projection to the last term of the fibered product and e_+ is the endpoint map. In this case, the map σ_P can be described geometrically as the endpoint map of every piecewise gradient flow line. We say that an element $\sigma \in S_p^\infty(\hat{B}_i)$ has Morse-Bott degree $p+i$.

The next step is defining the intermediate complex $(\tilde{C}_\bullet(f), \partial)$. We defined $\tilde{C}_k$ to be the set of all smooth singular chains with Morse-Bott degree k. That is,

$$\tilde{C}_k(f) = \bigoplus_{i=0}^{k} S_{k-i}^\infty(\hat{B}_i).$$

The definition of ∂ is as follows: We define $\partial_0 : S_p^\infty(\hat{B}_i) \to S_{p-1}^\infty(B_i)$ to be $(-1)^{p+i}$ times the boundary operator defined in Section 2.2. For $1 \leq j \leq k$, we define $\partial_j(\sigma_P) = \sigma_R$, where $R = P \times_{\sigma_P, \hat{B}_i} \overline{\mathcal{M}}(\hat{B}_i, \hat{B}_j)$. If $j > i$, then ∂_j is defined to be the zero map. The maps ∂_j are well-defined by Lemma 2.30.

The boundary operator ∂ is defined by

$$\partial := \bigoplus_{j=0}^{m} : \partial_j : \tilde{C}_k(f) \to \tilde{C}_{k-1}(f).$$

Lemma 2.31 states that $(\tilde{C}_\bullet(f), \partial)$ is a chain complex. However, we want to identify $\partial_j(\sigma_P)$ with a smooth singular chain

$$\sigma = \sum_\alpha n_\alpha \sigma_\alpha \in S_{p+j-1}^\infty(\hat{B}_{i-j})$$

where all $\sigma_\alpha : P_\alpha \to \hat{B}_{i-j}$ are maps from faces of I^N to $\hat{B}_{i-j}$. To do so, we define a free abelian subgroup $D_p^\infty(\hat{B}_i) \subset S_p^\infty(\hat{B}_i)$ generated by the degeneracy conditions in Definition 2.34.

Definition 2.37 We define

$$C_k(\hat{B}_i) = S_k^\infty(\hat{B}_i)/D_k^\infty(\hat{B}_i)$$

to be the subgroup of *non-degenerate* smooth singular chains in $S_k^\infty(\hat{B}_i)$.

Definition 2.38 We define $C_k(f)$ to be the set of all non-degenerate smooth singular chains with Morse-Bott degree k. That is,

$$C_k(f) := \bigoplus_{i=0}^{k} C_{k-i}(\hat{B}_i).$$

The boundary map

$$\partial : C_k(f) \to C_{k-1}(f)$$

is the induced homomorphism from $\partial : \tilde{C}_k(f) \to \tilde{C}_{k-1}(f)$. The pair $(C_\bullet(f), \partial)$ is called the *Morse-Bott chain complex* of f.

Lemma 2.35 shows that

$$\partial : C_k(f) \to C_{k-1}(f)$$

is well-defined and that $\partial^2 : C_k(f) \to C_{k-2}(f)$ is the zero map (because it is induced from a zero map). Therefore, $(C_\bullet(f), \partial)$ is a chain complex.

Definition 2.39 The *Morse-Bott homology* of f are the groups

$$H_k(C_\bullet(f), \partial).$$

When $m = 2$, the complex can be pictured as follows:

$$\begin{tikzcd}
S_0^\infty(\hat{B}_2)/D_0^\infty(\hat{B}_2) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] \arrow[ddddr, "\partial_2"'] & 0 & & \\
\oplus & \oplus & & \\
S_1^\infty(\hat{B}_1)/D_1^\infty(\hat{B}_1) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & S_0^\infty(\hat{B}_1)/D_0^\infty(\hat{B}_1) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & 0 & \\
\oplus & \oplus & \oplus & \\
S_2^\infty(\hat{B}_0)/D_2^\infty(\hat{B}_0) \arrow[r, "\partial_0"] & S_1^\infty(\hat{B}_0)/D_1^\infty(\hat{B}_0) \arrow[r, "\partial_0"] & S_0^\infty(\hat{B}_0)/D_0^\infty(\hat{B}_0) \arrow[r, "\partial_0"] & 0 \\
\| & \| & \| & \\
C_2(f) \arrow[r, "\partial"] & C_1(f) \arrow[r, "\partial"] & C_0(f) \arrow[r, "\partial"] & 0
\end{tikzcd}$$

Examples

Example 2.40 Let $f : M \to \mathbb{R}$ be a constant function.

In this case, there is only one critical submanifold, which has index 0. That is, $\hat{B}_0 = M$ and $\hat{B}_i = \emptyset$ for $i > 0$. Then $\tilde{C}_k(f) = S_k^\infty(\hat{B}_0)$ and degeneracy conditions (3)-(5) in Definition 2.34 are vacuus since there are no fibered products. Since conditions (1) and (2) in definitions 2.10 and 2.34 are the same, we get that $S_k(\hat{B}_0)$ and $D_k(\hat{B}_0)$ are both the subgroups restricted to smooth singular chains of $S_k(\hat{B}_0)$ and $D_k(\hat{B}_0)$ respectively. The boundary operator ∂_0 is defined to be $(-1)^k$ times the boundary operator in Section 2.2, which does not affect the homology.

The complex looks like (for $m = 2$):

$$
\begin{tikzcd}
0 \arrow[r, "\partial_0"] \arrow[dd, phantom, "\oplus"] \arrow[rdd, "\partial_1"] \arrow[rdddd, "\partial_2"'] & 0 \arrow[dd, phantom, "\oplus"] & & \\
& & & \\
0 \arrow[r, "\partial_0"] \arrow[dd, phantom, "\oplus"] \arrow[rdd, "\partial_1"] & 0 \arrow[r, "\partial_0"] \arrow[dd, phantom, "\oplus"] \arrow[rdd, "\partial_1"] & 0 \arrow[dd, phantom, "\oplus"] & \\
& & & \\
S_2^\infty(\hat{B}_0)/D_2^\infty(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[d, phantom, "\parallel"] & S_1^\infty(\hat{B}_0)/D_1^\infty(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[d, phantom, "\parallel"] & S_0^\infty(\hat{B}_0)/D_0^\infty(\hat{B}_0) \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
C_2(f) \arrow[r, "\partial"] & C_1(f) \arrow[r, "\partial"] & C_0(f) \arrow[r, "\partial"] & 0
\end{tikzcd}
$$

The bottom row is equivalent to the smooth singular N-cube chain complex $(S_k^\infty(\hat{B}_0)/D_k^\infty(\hat{B}_0), \partial)$ from Remark 2.12.

Therefore,

$$H_k(C_k(f), \partial) \cong H_k(S_k^\infty(\hat{B}_0)/D_k^\infty(\hat{B}_0), \partial).$$

By Theorem 2.11 and Remark 2.12,

$$H_k(S_k^\infty(\hat{B}_0)/D_k^\infty(\hat{B}_0), \partial) \cong H_k(\hat{B}_0; \mathbb{Z}).$$

And since $\hat{B}_0 = M$,

$$H_k(C_\bullet(f), \partial) \cong H_k(M; \mathbb{Z})$$

so the chain complex $(C_\bullet(f), \partial)$ yields the same homology as the singular homology.

Example 2.41 Let $f : M \to \mathbb{R}$ be a Morse-Smale function.

Since every critical point is isolated, $\hat{B}_i$ is a zero-dimensional submanifold of M, and hence a finite set of points. Hence, every singular C_p space

$$\sigma : P \to \hat{B}_i$$

is a constant map (since P is connected). If P is not zero-dimensional face of I^N, then σ does not depend on a free coordinate of P and hence $\sigma \in D_p^\infty(\hat{B}_i)$. In addition, by the fourth degeneracy condition, $\sigma_R \in D_k^\infty(\hat{B}_i)$ if $k > 0$ and R is a connected component of a fibered product. Therefore,

$$D_k^\infty(\hat{B}_i) = S_k^\infty(\hat{B}_i)$$

for every $k = 1, \ldots, m$, and

$$\partial_j : C_k(\hat{B}_i) \to C_{k+j-1}(\hat{B}_{i-j})$$

is trivial if $j \neq 1$. The first degeneracy condition identfies between all maps with the same image and hence implies that $S_0^\infty(\hat{B}_i)/D_0^\infty(\hat{B}_i)$ is the free abelian group generated by the points in $\hat{B}_i$.

If $\dim M = 2$, the complex can be pictured as follows:

$$\begin{tikzcd}
S_0^\infty(\hat{B}_2)/D_0^\infty(\hat{B}_2) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] \arrow[ddddr, "\partial_2"'] & 0 & & \\
\oplus & \oplus & & \\
0 \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & S_0^\infty(\hat{B}_0)/D_0^\infty(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & 0 & \\
\oplus & \oplus & \oplus & \\
0 \arrow[r, "\partial_0"] & 0 \arrow[r, "\partial_0"] & S_0^\infty(\hat{B}_0)/D_0^\infty(\hat{B}_0) \arrow[r] & 0 \\
\| & \| & \| & \\
C_2(f) \arrow[r, "\partial"] & C_1(f) \arrow[r, "\partial"] & C_0(f) \arrow[r, "\partial"] & 0
\end{tikzcd}$$

If $q \in \hat{B}_i$, then

$$\partial_1(q) : q \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-1}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-1}) \xrightarrow{e_+} B_{i-1}$$

counts the signed bumber of gradient flow lines from q to B_{i-1}, so

$$\partial_1(q) = \sum_{p \in \hat{B}_{i-1}} n(q,p)p \in C_{k-1}(f) = \hat{B}_{i-1}.$$

Hence, we get that $(C_\bullet(f), \partial)$ is the Morse-Smale complex, and since the Morse homology is isomorphic to the singular homology [4, Theorem 4.9.3],

$$H_k(C_\bullet(f), \partial) = H_k(M; \mathbb{Z}).$$

Example 2.42 Let M be $S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$ and $f(x,y,z) = z^2$. We are going to nuse the standard metric on S^2.

Then $\hat{B}_0 = S^1 \times \{0\}$, $\hat{B}_1 = \emptyset$ and $\hat{B}_2 = \{N, S\}$ where N, S are the north and south pole, respectively. Similarly to Example 2.41, the first and second degeneracy conditions imply that $S_0^\infty(\hat{B}_2)/D_0^\infty(\hat{B}_2) \cong \langle N, S \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ and $S_k^\infty(\hat{B}_2)/D_k^\infty(\hat{B}_2) = 0$ if $k > 0$.

The complex can be pictured as:

$$
\begin{tikzcd}
\langle N, S \rangle \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] \arrow[ddddr, "\partial_2"'] & 0 & & \\
\oplus & \oplus & & \\
0 \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & 0 \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & 0 & \\
\oplus & \oplus & \oplus & \\
S_2^\infty(\hat{B}_0)/D_2^\infty(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[d, equal] & S_1^\infty(\hat{B}_0)/D_1^\infty(\hat{B}_0) \arrow[r, "\partial_0"] \arrow[d, equal] & S_0^\infty(\hat{B}_0)/D_0^\infty(\hat{B}_0) \arrow[r] \arrow[d, equal] & 0 \\
C_2(f) \arrow[r, "\partial"] & C_1(f) \arrow[r, "\partial"] & C_0(f) \arrow[r, "\partial"] & 0
\end{tikzcd}
$$

The bottom row is the smooth N-cube chain complex $C_k(\hat{B}_0, \partial_0)$, and

$$H_k(S_k^\infty(\hat{B}_0)/D_k^\infty(\hat{B}_0), \partial_0) \cong H_k(\hat{B}_0; \mathbb{Z})$$

by Theorem 2.11 and Remark 2.12. The moduli space

$$\overline{\mathcal{M}}(\hat{B}_2, \hat{B}_0) = \mathcal{M}(\hat{B}_2, \hat{B}_0)$$

is a disjoint union of two copies of S^1 with opposite orientation, namely

$$\overline{\mathcal{M}}(\hat{B}_2, \hat{B}_0) = N \times_N \overline{\mathcal{M}}(\hat{B}_2, \hat{B}_0) \cup S \times_S \overline{\mathcal{M}}(\hat{B}_2, \hat{B}_0).$$

Therefore, there is an orientation reversing map

$$\alpha : N \times_N \overline{\mathcal{M}}(\hat{B}_2, \hat{B}_0) \to S \times_S \overline{\mathcal{M}}(\hat{B}_2, \hat{B}_0)$$

such that $\partial_2(N) \circ \alpha = \partial_0(S)$. Also, $\partial_0(\partial_2(N)) = \partial_0(\partial_2(S)) = 0$. The third condition of degeneracy in Definition 2.34 implies that

$$\partial_2(N + S) = \partial_2(N) + \partial_2(S) \in D_1^\infty(\hat{B}_0).$$

Using the fact that $|z|$ is strictly decreasing along flow lines, we can identify

$$R := N \times_N \overline{\mathcal{M}}(\hat{B}_2, \hat{B}_0) \cong S^1 \times \{1/2\}.$$

Now, let $\sigma_\alpha \in S_1(R)$ be a smooth singular chain in R that represents the fundamental class of R (that is, a generator of $H_1(R, \partial R)$). Since R is a closed manifold, $\partial\sigma_\alpha = 0$ and so $\partial_0\sigma_R = \partial_0(\sigma_R \circ \sigma_\alpha) = 0$. By the fifth condition of degeneracy in Definition 2.34,

$$\sigma_R - \sigma_R \circ \sigma_\alpha \in D_1^\infty(\hat{B}_0).$$

Observe that $\partial_R(x, y, 1/2) = (x, y, 0)$ and hence ∂_R is a diffeomorphism between R and $\hat{B}_0$, $\sigma_R \circ \sigma_\alpha$ represents the fundamental class of $\hat{B}_0$. This means that

$$\mathrm{Im}\,(\partial : C_2(f) \to C_1(f)) = \ker\,(\partial : C_1(f) \to C_0(f))$$

and hence $H_1(C_\bullet(f), \partial) = 0$. In addition,

$$H_2(C_\bullet(f), \partial) = \frac{\langle N, S\rangle}{\langle N + S\rangle} \cong \mathbb{Z}$$

and hence

$$H_k\,(C_\bullet(f), \partial) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & k \neq 0, 2 \end{cases}$$

Example 2.43 Let M be S^2 as in Example 2.42 and

$$f(x, y, z) = -z^2.$$

This time $\hat{B}_0 = \{N, S\}$ and $\hat{B}_1 = S^1 \times \{0\}$. By the same argument as in examples 2.41 and 2.42,

$$S_0^\infty(\hat{B}_0)/D_0^\infty(\hat{B}_0) \cong \langle N, S\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

and $S_k^\infty(\hat{B}_0)/D_k^\infty(\hat{B}_0) = 0$ if $k > 0$.

The Morse-Bott complex looks like:

$$\begin{tikzcd}
S_1^\infty(\hat{B}_1)/D_1^\infty(\hat{B}_1) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & S_0^\infty(\hat{B}_1)/D_0^\infty(\hat{B}_1) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_1"] & 0 & \\
\oplus & \oplus & \oplus & \\
0 \arrow[r, "\partial_0"] & 0 \arrow[r, "\partial_0"] & \langle N, S\rangle \arrow[r] & 0 \\
\| & \| & \| & \\
C_2(f) \arrow[r, "\partial"] & C_1(f) \arrow[r, "\partial"] & C_0(f) \arrow[r, "\partial"] & 0
\end{tikzcd}$$

The second row is the chain complex $C_k(\hat{B}_1, \partial_0)$. As in the previous example, $\overline{\mathcal{M}}(\hat{B}_1, \hat{B}_0)$ consists of two copies of S^1 with opposite orientation. We can see from

$$\partial_1(\sigma_P) : P \times_{\hat{B}_1} \overline{\mathcal{M}}(\hat{B}_1, \hat{B}_0) \xrightarrow{\pi_2} \overline{\mathcal{M}}(\hat{B}_1, \hat{B}_0) \xrightarrow{e_+} \hat{B}_0 = \{N, S\}$$

that $\partial_1(\sigma_P) \in \{N - S, S - N\}$ (Depending on the orientation). This means that

$$\ker\,(\partial : C_1(f) \to C_0(f)) = 0$$

and hence $H_1(C_\bullet(f), \partial) = 0$. In addition,

$$H_0(f) = \frac{\langle N, S\rangle}{\langle N - S\rangle} \cong \mathbb{Z}$$

and hence,

$$H_k(C_\bullet(f), \partial) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & k \neq 0, 2 \end{cases},$$

2.4 Independence On The Function

In this section, we prove that the homology the chain complex $(C_\bullet(f), \partial)$ does not depend on the choice of the function f. In the previous section, we have seen that if f is constant, the homology of $(C_\bullet(f), \partial)$ is the singular homology. Therefore, $H_k(C_k(f), \partial) \cong H_k(M; \mathbb{Z})$ for every f.

We are going to prove it similarly to [1, Theorem 6.17], but using the outline of the proof for [4, Theorem 3.4.2]. The proofs share very similar ideas, which can also be found in [11, Theorem 3.1], [26, Theorem 1.1], [27, Chapitre 1], and [28, Theorem 8].

Theorem 2.44 *Let $f_0, f_1 : M \to \mathbb{R}$ be two Morse-Bott-Smale functions. Then*

$$H_k(C_\bullet(f_0), \partial) \cong H_k(C_\bullet(f_1), \partial)$$

for every $k \in \mathbb{N}$. In particular, $H_k(C_\bullet(f_1), \partial) \cong H_k(M; \mathbb{Z})$.

The outline of the proof is as follows: We choose a smooth function

$$\begin{aligned} F : M \times [0,1] &\to \mathbb{R} \\ (x, s) &\mapsto F_s(x) = F(x, s) \end{aligned}$$

which satisfies

$$\begin{cases} F_s(X) = f_0(X) & s \leq \frac{1}{3} \\ F_s(X) = f_1(X) & s \geq \frac{2}{3} \end{cases}$$

1. We deduce a chain morphism

$$\Phi^F : (C_\bullet(f_0), \partial_{f_0}) \to (C_\bullet(f_1), \partial_{f_1}).$$

2. We show that if $f_0 = f_1$ and $F_s(x) = f_0(x) = f_1(x)$ then Φ^F is the identity morphism.

3. We show that if $f_2 : M \to \mathbb{R}$ is another Morse-Bott function, $F : M \times \mathbb{R} \to \mathbb{R}$ is a homotopy from f_0 to f_1, $G : M \times \mathbb{R} \to \mathbb{R}$ is a homotopy from f_1 to f_2 and $H : M \times \mathbb{R} \to \mathbb{R}$ is a homotopy from f_0 to f_2 so that F, G, H all satisfy the conditions in 2.4, then

$$\Phi^G \circ \Phi^F, \varphi^H : (C_\bullet(f_0), \partial_{f_0}) \to (C_\bullet(f_2), \partial_{f_2})$$

induce the same map on homology.

Using those properties, we can choose $f_2 = f_0$ and $H(s,x) = f_0(x)$ (the constant homotopy). Then $\Phi^G = (\Phi^F)^{-1}$ on homology and so Φ^F is an isomorphism on homology.

Note: The chain complex depends not only on the function but also on the Riemannian metric. However, the independence of the metric follows from the fact that the homology of a constant function (Example 2.40) does not depend on the metric.

Proof (First Step) We extend F to $M \times \left[-\frac{1}{3}, \frac{4}{3}\right]$ by setting $F_s(x) = f_0(x)$ for $s < 0$ and $F_s(x) = f_1(x)$ for $s > 1$.

Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = C \cdot (2x^3 - 3x^2)$ for some $C > 0$. g is a Morse function with local maximum at 0 and a local minimum at 1. If C is sufficiently large, then $\frac{\partial F}{\partial s}(x,s) + g'(s) < 0$ for every $(x,s) \in M \times \mathbb{R}$. The function $\tilde{F} = F + g$ is a Morse-Bott function whose critical points are

$$\mathrm{Crit}(\tilde{F}) = \mathrm{Crit}(f_0) \times \{0\} \cup \mathrm{Crit}(f_1) \times \{1\}.$$

For $B \subset \mathrm{Crit}(f_0)$ a critical submanifold of f_0,

$$\lambda^{\tilde{F}}_{B \times \{0\}} = \lambda^{f_0}_B + 1$$

and for $B' \subset \mathrm{Crit}(f_1)$ a critical submanifold of f_1,

$$\lambda^{\tilde{F}}_{B' \times \{1\}} = \lambda^{f_1}_{B'}.$$

By perturbing F a little in the range $M \times \left[\frac{1}{3}, \frac{2}{3}\right]$, we can assume that $W^u(B \times \{0\})$ and $W^s(B' \times \{1\})$ intersect transversely for all $B \subset \mathrm{Crit}(f_0)$ and $B' \subset \mathrm{Crit}(f_1)$ critical submanifolds of f_0, f_1 respectively. This is a weaker requirement than f being Morse-Bott-Smale.

We denote $\hat{B}^0_i, \hat{B}^1_i$ the set of all critical points with index i of f_0 and f_1, respectively. For critical submanifolds $B^0_i \subset \hat{B}^0_i, B^1_i \subset \hat{B}^1_i$, we define

$$\mathcal{M}_F(B^0_i, B^1_j) := \left(W^u(B^0_i) \pitchfork W^s(B^1_j)\right)/\mathbb{R}.$$

$\mathcal{M}_F(B^0_i, B^1_j)$ is a quotient of a manifold by a free $\mathbb{R}$-action, and hence a manifold of dimension

$$\lambda^{\tilde{F}}_{B^0_i} - \lambda^{\tilde{F}}_{B^1_j} - 1 = i + 1 - j - 1 = i - j.$$

Theorem 2.19 can be applied for $\mathcal{M}_F(B^0_i, B^1_j)$ as well, giving a compactification $\overline{\mathcal{M}}_F(B^0_i, B^1_j)$ of $\mathcal{M}_F(B^0_i, B^1_j)$. We set the *degree* of $\mathcal{M}_F(B^0_i, B^1_j)$ to be $i - j$. The boundary operator d (in a setting analoguous to Definition 2.20) is defined on $\overline{\mathcal{M}}_F(B_i^0, B_j^1)$ by

$$d\left(\overline{\mathcal{M}}_F(B_i^0, B_j^1)\right) = (-1)^{b_i^0+i}\left(\sum_{k=1}^{i-1}\overline{\mathcal{M}}_F(B_i^0, \hat{B}_k^0, B_j^1) - \sum_{k=j+1}^{m}\overline{\mathcal{M}}_F(B_i^0, \hat{B}_k^1, B_j^1)\right).$$

Similarly to Lemma 2.16, d decreases the degree of $\mathcal{M}_F(B_i^0, B_j^1)$ by 1.

We claim that $d^2\left(\overline{\mathcal{M}}_F(B_i^0, B_j^1)\right) = 0$. To show that, we first compute

$$\begin{aligned}
d\left(\overline{\mathcal{M}}_F(B_i^0, B_k^0, B_j^1)\right) &= d\left(\overline{\mathcal{M}}_F(B_i^0, B_k^0) \times_{B_k^0} \overline{\mathcal{M}}_F(B_k^0, B_j^1)\right) \\
&= (-1)^{i+b_i^0}\sum_{k<s<i}\sum_{B_s^0}\overline{\mathcal{M}}_F(B_i^0, B_s^0, B_k^0, B_j^1) \\
&\quad + (-1)^{i-k+b_i^0-1+b_k^0}\sum_{s=0}^{k-1}\sum_{B_s^0}(-1)^{k+b_k^0}\overline{\mathcal{M}}(B_i^0, B_k^0, B_s^0, B_j^1) \\
&\quad - (-1)^{i-k+b_i^0-1+b_k^0}\sum_{s=j+1}^{m}\sum_{B_s^1}(-1)^{k+b_k^0}\overline{\mathcal{M}}(B_i^0, B_k^0, B_s^1, B_j^1) \\
&= (-1)^{i+b_i^0}\left(\sum_{k<s<i}\sum_{B_s^0}\overline{\mathcal{M}}_F(B_i^0, B_s^0, B_k^0, B_j^1)\right. \\
&\qquad - \sum_{s=0}^{k-1}\sum_{B_s^0}\overline{\mathcal{M}}(B_i^0, B_k^0, B_s^0, B_j^1) \\
&\qquad \left. + \sum_{s=j+1}^{m}\sum_{B_s^1}\overline{\mathcal{M}}(B_i^0, B_k^0, B_s^1, B_j^1)\right) \\
&= (-1)^{i+b_i^0}\left(\sum_{k<s<i}\overline{\mathcal{M}}_F(B_i^0, \hat{B}_s^0, B_k^0, B_j^1)\right. \\
&\qquad - \sum_{s=0}^{k-1}\overline{\mathcal{M}}(B_i^0, B_k^0, \hat{B}_s^0, B_j^1) \\
&\qquad \left. + \sum_{s=j+1}^{m}\overline{\mathcal{M}}(B_i^0, B_k^0, \hat{B}_s^1, B_j^1)\right)
\end{aligned}$$

where the sum B_s^l runs over all critical submanifolds $B_s^l \subset \hat{B}_s^l$ $(l = 0, 1)$.

Similarly,

$$d\left(\overline{\mathcal{M}}_F(B_i^0, B_k^1, B_j^1)\right) = (-1)^{i+b_i^0}\left(\sum_{s<i}\overline{\mathcal{M}}_F(B_i^0, \hat{B}_s^0, B_k^1, B_j^1)\right.$$

$$- \sum_{k+1}^{m} \overline{\mathcal{M}}(B_i^0, \hat{B}_s^1, B_k^1, B_j^1)$$
$$+ \sum_{s=j+1}^{k-1} \overline{\mathcal{M}}(B_i^0, B_k^1, \hat{B}_s^1, B_j^1) \Bigg).$$

Summing everything yields:

$$\begin{aligned}
d^2\left(\overline{\mathcal{M}}_F(B_i^0, B_j^1)\right) &= (-1)^{i+b_i^0} d \left(\sum_{k=1}^{i-1} \overline{\mathcal{M}}_F(B_i^0, \hat{B}_k^0, B_j^1) - \sum_{k=j}^{m} \overline{\mathcal{M}}_F(B_i^0, \hat{B}_k^1, B_j^1) \right) \\
&= \sum_{k=0}^{i-1} \sum_{s=0}^{k-1} \left[\overline{\mathcal{M}}_F(B_i^0, \hat{B}_s^0, \hat{B}_k^0, B_j^1) - \overline{\mathcal{M}}_F(B_i^0, \hat{B}_s^0, \hat{B}_k^0, B_j^1) \right] \\
&\quad + \sum_{k=0}^{i-1} \sum_{s=j+1}^{m} \left[\overline{\mathcal{M}}_F(B_i^0, \hat{B}_k^0, \hat{B}_s^1, B_j^1) - \overline{\mathcal{M}}_F(B_i^0, \hat{B}_k^0, \hat{B}_s^1, B_j^1) \right] \\
&\quad + \sum_{k=j+1}^{m} \sum_{s=j+1}^{k-1} \left[\overline{\mathcal{M}}_F(B_i^0, \hat{B}_s^1, \hat{B}_k^1, B_j^1) - \overline{\mathcal{M}}_F(B_i^0, \hat{B}_s^1, \hat{B}_k^1, B_j^1) \right] \\
&= 0
\end{aligned}$$

We define $C_\bullet(F) := C_\bullet(f_0) \oplus C_{\bullet+1}(f_1)$ and

$$\partial_F : C_k(\tilde{F}) \to C_{k-1}(\tilde{F})$$

by

$$\partial_F = \begin{pmatrix} -\partial_{f_0} & \Phi^F \\ 0 & \partial_{f_1} \end{pmatrix}$$

where

$$\Phi^F : C_k(f_0) \to C_k(f_1)$$

is defined on a generator $\sigma_P : P \to \hat{B}_i^0$ as follows:

We write $R_j := P \times_{\hat{B}_i^0} \overline{\mathcal{M}}_F(\hat{B}_i^0, \hat{B}_j^1)$ and define

$$\Phi^F(\sigma_P) = \sum_{j=0}^{m} \sigma_{R_j}$$

and extended linearly. The identification between σ_{R_j} and $\sigma \in C_k(f_1)$ is defined using the degeneracy conditions in Definition 2.34. In addition, there is such an identification by Lemma 2.36.

To show that $\partial_{f_1} \circ \Phi^F = \Phi^F \circ \partial_{f_0}$, it is sufficient to show on generators. We compute both sides. To simplify the notation, we identify between an abstract topological chain and its domein. This is allowed because there is only one map from every domain in the following computations.

Let

$$\sigma_P : P \to \hat{B}_i^0 \in C_p(\hat{B}_i).$$

We calculate $\varphi^F \circ \partial_{f_0}(\sigma_P)$:

$$\begin{aligned}
\Phi^F(\partial_{f_0}(P)) &= \Phi^F\left(\partial_0(\sigma_P) + \sum_{l=0}^{i-1} P \times_{\hat{B}_i^0} \overline{\mathcal{M}}(\hat{B}_i^0, \hat{B}_l^0)\right) \\
&= \sum_{j=0}^{m} \Bigg[(-1)^{p+i} d(P) \times_{\hat{B}_i^0} \overline{\mathcal{M}}(\hat{B}_i^0, \hat{B}_j^1) \\
&\qquad + \sum_{l=0}^{i-1} P \times_{\hat{B}_i^0} \overline{\mathcal{M}}(\hat{B}_i^0, \hat{B}_l^0, \hat{B}_j^1) \Bigg].
\end{aligned}$$

Now, we calculate $\partial_{f_1} \circ \Phi^F(\sigma_P)$. Recall that $\partial_{f_1} = \bigoplus_{j=0}^{m} \partial_j$. We separate the cases $j = 0$ and $j \neq 0$.

$$\begin{aligned}
\partial_0(\Phi^F(\sigma_P)) &= (-1)^{p+i}\left(\sum_{j=0}^{m} P \times_{\hat{B}_i^0} \overline{\mathcal{M}}(\hat{B}_i^0, \hat{B}_j^1)\right) \\
&= (-1)^{p+i}\Bigg[\sum_{j=0}^{m} d(P) \times_{\hat{B}_i^0} \overline{\mathcal{M}}(\hat{B}_i^0, \hat{B}_j^1) \\
&\qquad + \sum_{B_i} (-1)^{p+b_i} P \times_{B_i} d\left(\overline{\mathcal{M}}(B_i, \hat{B}_j^1)\right)\Bigg] \\
&= (-1)^{p+i}\Bigg[\sum_{j=0}^{m} d(P) \times_{\hat{B}_i^0} \overline{\mathcal{M}}(\hat{B}_i^0, \hat{B}_j^1) \\
&\qquad + \sum_{B_i} (-1)^{p+b_i} \sum_{l=0}^{i-1} (-1)^{i+b_i} P \times_{B_i} \overline{\mathcal{M}}(B_i, \hat{B}_l^0, \hat{B}_j^1) \\
&\qquad - \sum_{B_i} (-1)^{p+b_i} \sum_{l=j+1}^{m} (-1)^{i+b_i} P \times_{B_i} \overline{\mathcal{M}}(B_i, \hat{B}_l^1, \hat{B}_j^1)\Bigg] \\
&= (-1)^{p+i}\Bigg[\sum_{j=0}^{m} d(P) \times_{\hat{B}_i^0} \overline{\mathcal{M}}(\hat{B}_i^0, \hat{B}_j^1) \\
&\qquad + \sum_{B_i} \sum_{l=0}^{i-1} (-1)^{p+i} P \times_{B_i} \overline{\mathcal{M}}(B_i, \hat{B}_l^0, \hat{B}_j^1) \\
&\qquad - \sum_{B_i} \sum_{l=j+1}^{m} (-1)^{p+i} P \times_{B_i} \overline{\mathcal{M}}(B_i, \hat{B}_l^1, \hat{B}_j^1)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{p+i}\sum_{j=0}^{m} d(P)\times_{\hat{B}_i^0}\overline{\mathcal{M}}(\hat{B}_i^0,\hat{B}_j^1)\\
&\qquad + \sum_{l=0}^{m} P\times_{\hat{B}_i}\overline{\mathcal{M}}(\hat{B}_i,\hat{B}_l^0,\hat{B}_j^1) - \sum_{l=j+1}^{m} P\times_{\hat{B}_i}\overline{\mathcal{M}}(\hat{B}_i,\hat{B}_l^1,\hat{B}_j^1)
\end{aligned}
$$

where the sum $\sum_{B_i}$ runs over all critical submanifolds $B_i \in \hat{B}_i$. If $j \neq 0$, then

$$
\begin{aligned}
\partial_j(\Phi^F(\sigma_P)) &= \partial_j\left(\sum_{l=0}^{m} P\times_{\hat{B}_i^0}\overline{\mathcal{M}}(\hat{B}_i^0,\hat{B}_l^1)\right)\\
&= \sum_{l=0}^{m} P\times_{\hat{B}_i^0}\overline{\mathcal{M}}(\hat{B}_i^0,\hat{B}_l^1,\hat{B}_{l-j}^1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\partial_{f_1}(\Phi^F(\sigma_P)) &= \sum_{j=0}^{m}\partial_j\left(\Phi^F(\sigma_P)\right)\\
&= (-1)^{p+i}\sum_{j=0}^{m} d(P)\times_{\hat{B}_i^0}\overline{\mathcal{M}}(\hat{B}_i^0,\hat{B}_j^1)\\
&\qquad + \sum_{l=0}^{m} P\times_{\hat{B}_i}\overline{\mathcal{M}}(\hat{B}_i,\hat{B}_l^0,\hat{B}_j^1) - \sum_{l=j+1}^{m} P\times_{\hat{B}_i}\overline{\mathcal{M}}(\hat{B}_i,\hat{B}_l^1,\hat{B}_j^1)\\
&\qquad + \sum_{l=j+1}^{m} P\times_{\hat{B}_i}\overline{\mathcal{M}}(\hat{B}_i,\hat{B}_l^1,\hat{B}_j^1)\\
&= (-1)^{p+i}\sum_{j=0}^{m} d(P)\times_{\hat{B}_i^0}\overline{\mathcal{M}}(\hat{B}_i^0,\hat{B}_j^1)\\
&\qquad + \sum_{l=0}^{m} P\times_{\hat{B}_i}\overline{\mathcal{M}}(\hat{B}_i,\hat{B}_l^0,\hat{B}_j^1).
\end{aligned}
$$

Therefore

$$\Phi^F\circ(-\partial_{f_0}) + \partial_{f_1}\circ\Phi^F = \partial_{f_1}\circ\Phi^F - \Phi^F\circ\partial_{f_0} = 0$$

so Φ^F is a chain morphism from f_0 to f_1.

In addition,

$$\partial_{\tilde{F}}^2 = \begin{pmatrix}\partial_{f_0}^2 & \Phi^F\circ(-\partial_{f_0}) + \partial_{f_1}\circ\Phi^F\\ 0 & \partial_{f_1}^2\end{pmatrix} = 0$$

so ∂_F is a boundary map. In particular, $(C_\bullet(F),\partial_F)$ is a chain complex. □

Proof (Second Step) Let $\tilde{F}$ be as in the proof of the first step and let $B \subset M$ be a critical submanifold of f. Denote by $B^i := B \times \{i\}$ for $i = 0, 1$. Then for every $(x, s) \in M \times (0, 1)$ we have

$$e_-(x,s) = (x,0),\ e_+(x,s) = (x,1)$$

and therefore $\overline{\mathcal{M}}(B^0, B^1) = \mathcal{M}(B^0, B^1) = (B \times (0,1))/\mathbb{R} \cong B$. In addition, for every critical submanifold $B \neq B' \subset C$ we have $\overline{\mathcal{M}}(B^0, (B')^1) = \emptyset$ (since there is no flow line from B^0 to $(B')^1$). Hence for every $\sigma : P \to B \in C_k(B)$ we have

$$P \times_{B^0} \overline{\mathcal{M}}(B^0, B^1) \cong P$$

so $\partial_{\tilde{F}}(\sigma^0) = \sigma^1$ (where $\sigma^i = \sigma \times \{i\}$ for $i = 0, 1$). Hence $\partial_{\tilde{F}}$ is the identity morphism. □

Proof (Third Step) Let F be such a homotopy from f_0 to f_1, G a homotopy from f_1 to f_2 and H from f_0 to f_2. Let $K : M \times [-\frac{1}{3}, \frac{4}{3}]_s \times [-\frac{1}{3}, \frac{4}{3}]_t \to \mathbb{R}$ be as follows:

- $K_{s,t} = F_s$ for $-\frac{1}{3} \leq t \leq \frac{1}{3}$.
- $K_{s,t} = G_t$ for $\frac{2}{3} \leq s \leq \frac{4}{3}$.
- $K_{s,t} = H_t$ for $-\frac{1}{3} \leq t \leq \frac{1}{3}$.
- $K_{s,t} = f_2$ for $\frac{2}{3} \leq s \leq \frac{4}{3}$.

Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = C \cdot (2x^3 - 3x^2)$ for some $C > 0$ (as in the proof of the first step). For C sufficiently large, we have that:

- $\frac{\partial K}{\partial s}(x,s,t) + g'(s) < 0$ for every $x, s, t \in M \times (0,1) \times [-\frac{1}{3}, \frac{4}{3}]$.
- $\frac{\partial K}{\partial t}(x,s,t) + g'(t) < 0$ for every $x, s, t \in M \times [-\frac{1}{3}, \frac{4}{3}] \times (0,1)$.

Therefore, we set $\tilde{K} : M \times \mathbb{R} \times \mathbb{R}$ to be $K(x,s,t) + g(s) + g(t)$. similarly to the first step, the critical points are

$$\begin{aligned}\mathrm{Crit}(\tilde{K}) = \mathrm{Crit}(f_0) \times (0,0) \cup \mathrm{Crit}(f_1) \times (0,1) \\ \cup \mathrm{Crit}(f_2) \times (1,0) \cup \mathrm{Crit}(f_2) \times (1,1)\end{aligned}$$

and the indices are as follows:

- If $B \subset \mathrm{Crit}(f_0)$ is a critical submanifold, then $B' := B \times (0,0)$ is a critical submanifold of $\tilde{K}$ and $\lambda^{\tilde{K}}_{B'} = \lambda^{f_0}_B + 2$.
- If $B \subset \mathrm{Crit}(f_1)$ is a critical submanifold, then $B' := B \times (0,1)$ is a critical submanifold of $\tilde{K}$ and $\lambda^{\tilde{K}}_{B'} = \lambda^{f_1}_B + 1$.
- If $B \subset \mathrm{Crit}(f_0)$ is a critical submanifold, then $B' := B \times (1,0)$ and $B'' := B \times (1,0)$ are critical submanifolds of $\tilde{K}$ and $\lambda^{\tilde{K}}_{B'} = \lambda^{f_2}_B + 1$, $\lambda^{\tilde{K}}_{B''} = \lambda^{f_2}_B$.

As in the first step, using a little perturbation of $\tilde{K}$ where either $\frac{1}{3} < s < \frac{2}{3}$ or $\frac{1}{3} < t < \frac{2}{3}$ we can assume that $W^s(B)$ and $W^u(B')$ intersect transversely for every $B, B' \subset M$ critical submanifolds of $\tilde{K}$. Again, we can use 2.36 to identify

$$C_{\bullet+1}(\tilde{K}) = C_{\bullet-1}(f_0) \oplus C_\bullet(f_1) \oplus C_\bullet(f_2) \oplus C_{\bullet+1}(f_2).$$

We define $\partial_K : C_{\bullet+1}(\tilde{K})$ by using the first step twice, as it defines Morse-Bott homology on homotopies. Then, ∂_K has the matrix representation

$$\partial_K = \begin{pmatrix} \partial_{f_0} & 0 & 0 & 0 \\ -\Phi^F & -\partial_{f_1} & 0 & 0 \\ -\Phi^H & 0 & -\partial_{f_2} & 0 \\ \Phi^K & \Phi^G & id & \partial_{f_2} \end{pmatrix}$$

where $\varphi^F, \Phi^G, \Phi^H$ are defined as in the first step. The first step shows that $\partial_K^2 = 0$, which yields

$$\Phi^K \circ \partial_{f_0} - \Phi^G \circ \Phi^F - \Phi^H + \partial_{f_2} \circ \Phi^K = 0$$

or

$$\Phi^K \circ \partial_{f_0} + \partial_{f_2} \circ \Phi^K = \Phi^G \circ \Phi^F + \Phi^H$$

so $\Phi^G \circ \Phi^F$ and $-\Phi^H$ are chain-homotopic, and so induce the same map on homology (see [2, Proposition 2.12]). Since $-\Phi^H$ and Φ^H clearly induce the same map on homology, the result follows. □

2.5 Homology Over A Field

Let $\mathbb{F}$ be a field. We can define the Morse-Bott complex over $\mathbb{F}$ by replacing $C_k(f)$ with $C_k(f) \otimes \mathbb{F}$. That is, $C_k(f;\mathbb{F}) := C_k(f) \otimes \mathbb{F}$ and $\partial : C_k(f;\mathbb{F}) \to C_{k-1}(f;\mathbb{F})$ is defined on generators by

$$\sigma \otimes 1 \mapsto \partial\sigma \otimes 1$$

and extended linearly. Since $\partial^2(\sigma \otimes 1) = 0 \otimes 1 = 0$, $(C_k(f;\mathbb{F}))$ is indeed a chain complex.

Theorem 2.45 *Let $f_0, f_1 : M \to \mathbb{R}$ be two Morse-Bott-Smale functions. Then*

$$H_k(C_\bullet(f_0;\mathbb{F}), \partial) \cong H_k(C_\bullet(f_1;\mathbb{F}), \partial)$$

for every $k \in \mathbb{N}$.

Proof The proof is identical to the proof of Theorem 2.44. □

Remark 2.46 *Usually, the homology over a field is derived from the homology over $\mathbb{Z}$ using the universal coefficient theorem for homology [2, Theorem 3A.3]. However, the theorem requires the chain complex to be free, but the chain complex defined by Definition 2.34 is not necessarily free (see Theorem 3.17 for example).*

However, we can utilize the fact that the complex is the same as the Morse-Smale chain complex if f is Morse-Smale (and, in particular, free) to show that the Morse homology over a field is isomorphic to the singular homology. Since the homology is independent of the function, it is true for Morse-Bott functions as well.

Theorem 2.47 *Let $f : M \to \mathbb{R}$ be Morse-Bott-Smale. Then*

$$H_k(C_\bullet(f;\mathbb{F}), \partial) \cong H_k(M, \mathbb{F})$$

for every $k \in \mathbb{N}$.

Proof Assume first that f is Morse-Smale. In Example 2.41, we see that the chain complex is free and so we can use the universal coefficient theorem. By the universal coefficient theorem for homology [2, Theorem 3A.3], there is a short exact sequence

$$0 \to H_k(C_\bullet(f), \partial) \otimes \mathbb{F} \to H_k(C_\bullet(f;\mathbb{F}), \partial) \to \mathrm{Tor}(H_{n-1}(C_\bullet(f), \partial), \mathbb{F}) \to 0$$

and this sequence splits, so

$$H_k(C_\bullet(f;\mathbb{F}), \partial) \cong (H_k(C_\bullet(f), \partial) \otimes \mathbb{F}) \oplus \mathrm{Tor}(H_{n-1}(C_\bullet(f), \partial), \mathbb{F}).$$

On the other hand, the universal coefficient theorem also applies for $H_k(M)$, so

$$H_k(M;\mathbb{F}) \cong (H_k(M) \otimes \mathbb{F}) \oplus \mathrm{Tor}(H_{n-1}(M), \mathbb{F}).$$

Since $H_k(C_\bullet(f), \partial) \cong H_k(M)$ by Theorem 2.45, we get that $H_k(C_\bullet(f;\mathbb{F}), \partial) \cong H_k(M;\mathbb{F})$.

If f is not Morse-Smale, there is always a Morse-Smale function $g : M \to \mathbb{R}$ [7, Theorem 3.1]. Since $H_k(C_\bullet(f;\mathbb{F}), \partial) \cong H_k(C_\bullet(f;\mathbb{F}), \partial)$ and $H_k(f;\mathbb{F}) \cong H_k(g;\mathbb{F})$ by 2.45, the result follows. $\square$

We will be mostly interested in the case where $\mathbb{F} = \mathbb{Z}_2$.

Example 2.48 Let $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinate system induced from $\mathbb{R}^2$ and let $f : M \to \mathbb{R}$ be defined as $f([x,y]) = -\cos 2\pi x$.

The critical submanifolds of f are $\hat{B}_0 = \{[x,y] \in \mathbb{T}^2 \mid x = 0\} \cong S^1$ and $\hat{B}_1 = \{[x,y] \in \mathbb{T}^2 \mid x = \frac{1}{2}\} \cong S^1$

The complex can be pictured as:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_0"] & C_1(\hat{B}_1)\otimes\mathbb{Z}_2 \arrow[r, "\partial_0"] \arrow[dr, "\partial_1"] & C_0(\hat{B}_1)\otimes\mathbb{Z}_2 \arrow[r, "\partial_0"] \arrow[dr, "\partial_1"] & 0 & \\
& \oplus & \oplus & \oplus & \\
\cdots \arrow[r, "\partial_0"] & C_2(\hat{B}_0)\otimes\mathbb{Z}_2 \arrow[r, "\partial_0"] & C_1(\hat{B}_0)\otimes\mathbb{Z}_2 \arrow[r, "\partial_0"] & C_0(\hat{B}_0)\otimes\mathbb{Z}_2 \arrow[r, "\partial_0"] & 0 \\
& \| & \| & \| & \\
\cdots \arrow[r, "\partial_0"] & C_2(f;\mathbb{Z}_2) \arrow[r, "\partial"] & C_1(f;\mathbb{Z}_2) \arrow[r, "\partial"] & C_0(f;\mathbb{Z}_2) \arrow[r, "\partial"] & 0
\end{tikzcd}$$

Each row calculates calculates the homology of $\hat{B}_0 \cong \hat{B}_1 \cong S^1$. Also, $\overline{\mathcal{M}}(\hat{B}_1, \hat{B}_0) = \mathcal{M}(\hat{B}_1, \hat{B}_0)$ consists of two disjoint copies of S^1.

Using the metric induced from $\mathbb{R}^2$, we can see that

$$e_-(x,y) = (0,y),\ e_+(x,y) = (\frac{1}{2}, y)$$

for every $(x,y) \in \mathbb{T}^2 \setminus \mathrm{Crit}(f)$. Therefore, for every $(\sigma : P \to \hat{B}_1) \in C_p(\hat{B}_1)$, $P \times_{\hat{B}_1} \mathcal{M}(\hat{B}_1, \hat{B}_0)$ is diffeomorphic to two disjoint copies of P. Therefore $\partial_1(\sigma)$ consists of two identical maps, so $\partial_1(\sigma) = 0$. Hence,

$$H_k(C_\bullet(f,\mathbb{Z}_2),\partial) = \begin{cases} \mathbb{Z}_2 & k = 0,2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k = 1 \\ 0 & k \neq 0,1,2 \end{cases}.$$

Chapter 3

Orthogonal Groups

In this chapter we compute the homology groups of $SO(n)$ using the Morse-Bott system developed in Chapter 2. We are going to use the Morse-Bott function $f : SO(n) \to \mathbb{R}$, defined by $f(X) = X_{nn}$ (the lower-right coordinate). We show that the function is indeed Morse-Bott-Smale, with two critical submanifolds, F_0 and F_{n-1}, whose indices are 0 and $n-1$ respectively. Our main result is Theorem 3.18, which gives the short exact sequence

$$0 \to H_k(SO(2n-1)) \to H_k(SO(2n)) \to H_{k-2n+1}(SO(2n-1)) \to 0.$$

In addition, Theorem 3.23 gives the following recursive formula for the homology groups over $\mathbb{Z}_2$ of $SO(n)$:

$$H_k(SO(n);\mathbb{Z}_2) \cong H_k(SO(n-1);\mathbb{Z}_2) \oplus H_{k-n+1}(SO(n-1);\mathbb{Z}_2).$$

This recursive formula is also proved in [2, Theorem 3D.1] and [8, Section 5]. In addition, it can be derived from [18, Theorem 3].

We denote by $O(n)$ the group of all orthogonal matrices. That is,

$$O(n) := \left\{X \in M_{n\times n}(\mathbb{R}) \mid XX^t = I_n\right\}$$

where $M_{n\times n}(\mathbb{R})$ denotes all the $n \times n$ matrices over $\mathbb{R}$ and I_n is the identity matrix. Note that $O(n)$ is a Lie group of dimension $n(n-1)/2$ [29, Example 7.27]. Using the fact that

$$1 = \det(I_n) = \det(XX^t) = \det(X) \cdot \det(X^t) = \det(X)^2,$$

we get that $\det(X) = \pm 1$ for every $X \in O(n)$. We are interested in

$$SO(n) = \{X \in O(n) \mid \det(X) = 1\}$$

which is an open subset (and hence, a full-dimensional submanifold) of $O(n)$ because det is a Lie group homomorphism.

Let

$$GL(n,\mathbb{R}) = \{X \in M_{n\times n}(\mathbb{R}) \mid \det X \neq 0\}$$

be the set of all invertible matrices and let $\mathrm{Sym}(n,\mathbb{R})$ be the submanifold of $GL(n,\mathbb{R})$ consisting of all symmetric matrices. $\mathrm{Sym}(n,\mathbb{R})$ has $n(n+1)/2$ free coordinates and hence it is an $n(n+1)/2$-dimensional submanifold of $GL(n,\mathbb{R})$. Since $\left(AA^t\right)^t = \left(A^t\right)^t A^t = AA^t$, we have a well-defined smooth function

$$\psi : GL(n,\mathbb{R}) \to \mathrm{Sym}(n,\mathbb{R}), \quad \psi(X) = XX^t$$

and $O(n) = \psi^{-1}(I_n)$. The differential of ψ (as a function $GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$) is (using Leibniz rule)

$$d\psi_X(M) = XM^t + MX^t$$

and hence for $X \in SO(n)$,

$$T_X SO(n) = T_X O(n) = \ker d\psi_X = \{M \in M_{n\times n}(\mathbb{R}) \mid XM^t + MX^t = 0\}.$$

In particular, $T_{I_n} = \{M \in M_{n\times n}(\mathbb{R}) \mid M^t + M = 0\}$.

We are going to use the Riemannian metric on $SO(n)$ induced from $GL(n,\mathbb{R})$. That is, for $M, N \in T_X SO(n)$,

$$\langle M, N\rangle = \mathrm{tr}(M^t N) = \mathrm{tr}(MN^t) = \sum_{i,j=1}^{n} M_{ij}N_{ij}.$$

3.1 Linear functions on $SO(n)$

In this section, we discuss properties of linear functions on $SO(n)$. In particular, we give a condition for their critical points and compute the gradient of those functions. This section is based on Section 3 of [8].

We say that a function $f : SO(n) \to \mathbb{R}$ is *linear* if there exists a matrix $A \in M_{n\times n}$ so that $f(X) = f_A(X) = \mathrm{tr}(A^t X)$. The differential of a linear function f is (using linearity of the trace)

$$df_X(M) = \mathrm{tr}(A^t M).$$

The gradient of f is

$$\nabla f(X) = \frac{1}{2}(A - XA^t X). \tag{3.1}$$

Indeed,

$$X \cdot \nabla f(X)^t + \nabla f(X) \cdot X^t = \frac{1}{2}(XA^t - AX^t) + \frac{1}{2}(AX^t - XA^t) = 0$$

so $\nabla f(X) \in T_X SO(n)$ and for every $M \in T_X SO(n)$,

$$\begin{aligned}\langle \nabla f(X), M\rangle &= \mathrm{tr}(\frac{1}{2}(A - XA^tX)^tM) = \frac{1}{2}\mathrm{tr}(A^tM - X^tAX^tM) \\ &= \frac{1}{2}\mathrm{tr}(A^tM + X^tAM^tX) = \mathrm{tr}(A^tM) = df_X(M).\end{aligned}$$

The next lemma gives us some conditions for critical points of f.

Lemma 3.1 *The following are equivalent:*

1. $\nabla f(X) = 0$
2. $A^tX = X^tA$
3. A^tX *is symmetric.*

Proof

$$\nabla f(X) = \frac{1}{2}\left(A - XA^tX\right) = 0 \iff A = XA^tX \iff X^tA = A^tX.$$

This proves (1) $\iff$ (2). To see that (2) $\iff$ (3), observe that

$$(A^tX)^t = X^tA,$$

so $A^tX = X^tA$ if and only if A^tX is symmetric. □

Example 3.2 We consider the case $A = I_n$. Then $f(X) = \mathrm{tr}(X)$ and the critical points are the symmetric matrices in $SO(n)$.

Example 3.3 Let A be a diagonal matrix with n distinct eigenvalues. That is, $A_{ij} = 0$ and $A_{ii} \neq A_{jj}$ if $i \neq j$. Then, for every $X \in SO(n)$,

$$\left(A^tX\right)_{ij} = \sum_{k=1}^{n} \left(A^t\right)_{ik} X_{kj} = A_{ii}X_{ij}.$$

and similarly,

$$\left(X^tA\right)_{ij} = \sum_{k=1}^{n} \left(X^t\right)_{ik} A_{kj} = X_{ij}A_{jj}.$$

In particular, if $X \in \mathrm{Crit}(f)$, then using the above lemma, $A_{ii}X_{ij} = X_{ij}A_{jj}$ for every $1 \leq i, j \leq n$. This implies that $X_{ij} = 0$ if $i \neq j$ (since $A_{ii} \neq A_{jj}$). Therefore, the set of critical points is

$$\mathrm{Crit}(f) = \{X \in SO(n) \mid X_{ii} = \pm 1 \text{ for every } 1 \leq i \leq j\}$$

which is an discrete subset of $SO(n)$. It can be shown that f is a Morse function [18, Lemma 3] and that

$$\#\mathrm{Crit}_k(f) = \mathrm{rank} H_k(f; \mathbb{Z}_2)$$

for every $k \geq 0$. Therefore, the mod 2 homology of $SO(n)$ can be computed using this f [18, Theorem 3]. We later show another way to compute these homology groups (Theorem 3.23).

3.2 Mapping Cone

In this section we recall the construction of a mapping cone of two chain complexes (see Chapter 10 of [30] for a reference). Let $C_\bullet = (C_\bullet, \partial_C)$ and $D_\bullet = (D_\bullet, \partial_D)$ be chain complexes and $\varphi : C_\bullet \to D_\bullet$ be a map of chain complexes (i.e. $\varphi \circ \partial_C = \partial_D \circ \varphi$). The mapping cone of φ is the chain complex, $\text{cone}_\bullet = (C_{\bullet-1} \oplus D_\bullet, \partial)$ where

$$\partial(c, d) = (-\partial_C c, \partial_D d - \varphi(c)).$$

Lemma 3.4 $\text{cone}_\bullet$ *is a chain complex (i.e.* $\partial^2 : \text{cone}_\bullet \to \text{cone}_{\bullet-2}$ *is the zero map).*

Proof

$$\partial^2(c, d) = \partial(\partial_C c, \partial_D d - \varphi(c)) = (\partial_C^2 c, \partial_D^2 d - \partial_D \varphi(c) - \varphi(\partial_C c)).$$

Since $\partial_C^2(c) = 0$, $\partial_D^2(d) = 0$ and $\partial_D \varphi(c) - \varphi(\partial_C c) = 0$, we have $\partial^2(c, d) = 0$.$\square$

Using the fact that cone is also a chain complex, we get that

$$0 \to D_k \xrightarrow{i} \text{cone}_k \xrightarrow{j} C_{k-1} \to 0 \tag{3.2}$$

(where $i(d) = (0, d)$ and $j(c, d) = -c$) is a short exact sequence (SES). We claim that the following diagram is commutative:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial_D"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "-\partial_C"] & \\
0 \arrow[r] & D_k \arrow[r, "i"] \arrow[d, "\partial_D"] & \text{cone}_k \arrow[r, "j"] \arrow[d, "\partial"] & C_{k-1} \arrow[r] \arrow[d, "-\partial_C"] & 0 \\
0 \arrow[r] & D_{k-1} \arrow[r, "i"] \arrow[d, "\partial_D"] & \text{cone}_{k-1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{k-2} \arrow[r] \arrow[d, "-\partial_C"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Indeed, $\partial \circ i(d) = (0, \partial_D d) = i \circ \partial_D d$ and

$$j \circ \partial(c, d) = j(-\partial_C c, \partial_D d - \varphi(c)) = \partial_C c = -\partial_C \circ j(c, d).$$

Thus, Equation 3.2 is also an SES of chain complexes. By Theorem [2, Theorem 2.16], an SES of chain complexes can be extended to a long exact sequence (LES) of homology:

$$\cdots \to H_{k+1}(D) \to H_{k+1}(\text{cone}) \to H_k(C) \xrightarrow{\delta_*} H_k(D) \to \cdots$$

where $\delta_* : H_k(C) \to H_k(D)$ is the connecting homomorphism.

Lemma 3.5 $\delta_* = \varphi_* : H_k(C) \to H_k(D)$

Proof Let $[c] \in H_k(C)$. Then $j_*([-c, 0]) = [c]$, and

$$\partial_*([-c, 0]) = [-\partial_C c, \varphi(c)] = [0, \varphi(c)].$$

Now, $\varphi(c)$ is a cycle because c is a cycle, so $[\varphi(c)] \in H_k(D)$. Hence, by the definition of δ_* in [2, Theorem 2.16],

$$\delta_*([c]) = [\varphi(c)] = \varphi_*(c). \qquad \square$$

3.2.1 Simple Morse-Bott Functions

Let (M, g) be a closed m-dimensional Riemannian manifold and $f : M \to \mathbb{R}$ a Morse-Bott function. We say that f is *simple* if f has exactly two critical submanifolds, F_0 and F_n of indices 0 and n respectively (note that there is always a submanifold of index 0, as f always has global minima).

Since $W^s(F_0)$, the stable manifold of F_0, is full dimensional, $W^s(F_0)$ and $W^u(x)$ intersect transversely for every $x \in F_{n-1}$, and hence (f, X) is a Morse-Bott-Smale pair for every pseudo-gradient X (and in particular, for $X = -\nabla f$).

The complex of a simple Morse-Bott function f can be pictured as follows:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_0"] \arrow[ddr, "\partial_n"] & C_k(F_n) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_n"] & \cdots \arrow[r, "\partial_0"] \arrow[ddr, "\partial_n"] & C_0(F_n) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_n"] & 0 & \\
 & \oplus & \cdots & \oplus & \oplus & \\
\cdots \arrow[r, "\partial_0"] & C_{k+n}(F_0) \arrow[r, "\partial_0"] & \cdots \arrow[r, "\partial_0"] & C_n(F_0) \arrow[r, "\partial_0"] & C_{n-1}(F_0) \arrow[r, "\partial_0"] & \cdots \\
 & \| & \cdots & \| & \| & \\
\cdots \arrow[r, "\partial"] & C_{k+n}(f) \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & C_n(f) \arrow[r, "\partial"] & C_{n-1}(f) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

The chain complex $(C_\bullet(F_n), \partial_0)$ in the top row is identical up to a sign $(-1)^{k+n}$ to the smooth singular N-cube chain complex, $(S^\infty_\bullet(F_n)/D^\infty_\bullet(F_n), \partial)$, as defined in Subsection 2.2.3. Hence,

$$H_k(C_\bullet(F_n), \partial_0) = H_k(F_n)$$

and similarly,

$$H_k(C_\bullet(F_0), \partial_0) = H_k(F_0).$$

Let $\sigma \in C_k(F_n)$. By Lemma 2.31,

$$0 = \sum_{j=0}^{n} \partial_j(\partial_{n-j}(\sigma)) = \partial_0(\partial_n(\sigma)) + \partial_n(\partial_0(\sigma)).$$

Therefore, if we define $\widetilde{\partial_0} := -\partial_0 : C_k(F_n) \to C_{k-1}(F_n)$, we get that

$$\partial_n \circ \widetilde{\partial_0} = \partial_0 \circ \partial_n : C_k(F_n) \to C_{k+n-1}(F_0).$$

Thus,

$$\partial_n : \left(C_\bullet(F_n), \widetilde{\partial_0}\right) \to (C_{\bullet+n-1}(F_0), \partial_0)$$

is a chain map. Define the mapping cone

$$\text{cone} := \text{cone}(\partial_n) = C_{\bullet-n+1}(F_n) \oplus C_\bullet(F_0).$$

In this case, $C_\bullet = C_{\bullet-n+1}(F_n)$, $D_\bullet = C_\bullet(F_0)$ and $\varphi = -\partial_n$ Note that

$$\text{cone}_\bullet = C_{\bullet-n+1}(F_n) \oplus C_\bullet(F_0) = C_\bullet(f).$$

The map

$$\partial_{\text{cone}} : \text{cone}_\bullet \to \text{cone}_{\bullet-1}$$

is defined on generators by

$$\partial_{\text{cone}}(\sigma_P, \sigma_Q) = (-\widetilde{\partial_0}\sigma_P, \partial_0\sigma_Q + \partial_n\sigma_P) = (\partial_0\sigma_P, \partial_0\sigma_Q + \partial_n\sigma_P)$$

and extended linearly. The boundary operator ∂_{cone} has the same definition as $\partial : C_\bullet(f) \to C_{\bullet-1}(f)$ and hence $\partial_{\text{cone}} = \partial$. Therefore,

$$H_k(\text{cone}, \partial_{\text{cone}}) = H_k(C_\bullet(f), \partial)$$

and since $H_k(C_\bullet(f), \partial) \cong H_k(M)$ by Theorem 2.44,

$$H_k(\text{cone}, \partial_{\text{cone}}) = H_k(M).$$

Hence, we get the long exact sequence

$$\cdots \to H_k(F_0) \to H_k(M) \to H_{k-n}(F_n) \xrightarrow{(\partial_n)_*} H_{k-1}(F_0) \to \cdots . \tag{3.3}$$

3.3 A Morse-Bott function on $SO(n)$

In this section we look at the function $f : SO(n) \to \mathbb{R}$ defined by $f(X) = X_{nn}$ (the lower-right entry of X). To show that f is Morse-Bott, we write $f = p \circ h$ where $p : SO(n-1) \to S^{n-1}$ is the projection to the bottom row

$$p(X) = (X_{n1}, \ldots, X_{nn})$$

and $h : S^{n-1} \to \mathbb{R}$ is the height function $h(x_1, \ldots, x_n) = x_n$. It is well-known that h is Morse, and p is a submersion. The next lemma states that the composition is Morse-Bott.

Lemma 3.6 *Let M, N be manifolds of dimension m and n respectively. Let $g : M \to N$ be a submersion and $h : N \to \mathbb{R}$ a Morse function. Then the composition $f = g \circ h : M \to \mathbb{R}$ is Morse-Bott.*

Proof First, note that every $x \in N$ is a regular value of g. In particular, critical points of h are regular values of g. Therefore, $B := g^{-1}(x)$ is a submanifold for every critical point $x \in N$ of h.

Let $p \in B$. By the rank theorem [29, Theorem 4.12] there is a chart (U, φ) for M centered at $p \in M$ and a chart (V, ψ) for N centered at $g(p)$ so that g has a coordinate representation of the form

$$g(x_1, \ldots, x_n, \ldots x_m) = (x_1, \ldots, x_n).$$

Hence

$$df_p = dg_p \cdot dh_{g(p)} = (dh_{g(p)}, 0_{1\times(m-n)})$$

and

$$T_pB = \operatorname{span}\left\{\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_m}\right\}.$$

Therefore,

$$\operatorname{Hess}_p(f) = \begin{pmatrix} \operatorname{Hess}_{g(p)}(h) & 0 \\ 0 & 0_{(m-n)\times(m-n)} \end{pmatrix}.$$

and

$$\nu_pB = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}.$$

Therefore, the normal Hessian of f at p is

$$\operatorname{Hess}_p^{\nu}(f) = \left(\operatorname{Hess}_{g(p)}(h)\right).$$

which is non-degenerate because $\operatorname{Hess}_{g(p)}(h)$ is non-degenerate. Therefore, f is Morse-Bott. $\square$

Using Equation 3.1, we can write the $\nabla f(X)$ explicitly:

$$\nabla f(X) = \frac{1}{2}\begin{pmatrix} -X_{1n}X_{n1} & \ldots & -X_{1n}X_{n,n-1} & -X_{1n}X_{nn} \\ \vdots & \ddots & \vdots & \vdots \\ -X_{n-1,n}X_{n1} & \ldots & -X_{n-1,n}X_{n,n-1} & -X_{n-1,n}X_{nn} \\ -X_{nn}X_{n1} & \ldots & -X_{nn}X_{n,n-1} & 1 - X_{nn}X_{nn} \end{pmatrix}.$$

The next step is to find all critical points of f.

Lemma 3.7 $X \in \operatorname{Crit}(f)$ *if and only if* $X_{nn} = \pm 1$.

Proof If $X \in \mathrm{Crit}(f)$, then $(\nabla f(X))_{nn} = 1 - X_{nn}^2 = 0$. Therefore $X_n n = \pm 1$. On the other hand, if $X_{nn} = \pm 1$, then $\sum_{i=1}^n X_{in}^2 \geq X_{nn}^2 = 1$. But since $X \in SO(n)$, $\sum_{i=1}^n X_{in}^2 = 1$. Hence $X_{in} = 0$ if $i \neq n$. Similarly, $X_{ni} = 0$ if $i \neq n$. Therefore, $(\nabla f(X))_{ij} = 0$ if $(i,j) \neq (n,n)$ and $(\nabla f(X))_{nn} = 1 - X_{nn}^2 = 0$. Hence, $X \in \mathrm{Crit}(f)$. □

Using the above lemma, we can see that f has two critical submanifolds. Let $F_0 = \{X \in SO(n) \mid X_{nn} = -1\}$ and $F_{n-1} = \{X \in SO(n) \mid X_{nn} = 1\}$. The map $i : SO(n-1) \to F_{n-1}$ defined by

$$i(X) = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$$

is a diffeomorphism. Similarly, we can define a diffeomorphism

$$j : SO(n-1) \to F_0$$

by

$$j(X) = \begin{pmatrix} -X_{11} & \dots & -X_{1,n-1} & 0 \\ X_{21} & \dots & X_{2,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ X_{n-1,1} & \dots & X_{n-1,n-1} & 0 \\ 0 & \dots & 0 & -1 \end{pmatrix}.$$

Therefore, $F_0 \cong F_{n-1} \cong SO(n-1)$.

Since $f(X) \in [-1,1]$, the index of F_0 is 0. The index of F_{n-1} equals to

$$\dim(SO(n)) - \dim(F_{n-1}) = \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1.$$

Let $X \in SO(n)$. Since

$$\lim_{t \to \pm\infty} f(\varphi_t(X)) = \mp 1$$

and f is strictly decreasing along flow lines, there exists a unique $t \in \mathbb{R}$ so that $f(\varphi_t(X)) = 0$. In addition, the only moduli space is $\mathcal{M}(F_{n-1}, F_0)$ and so there are no broken flow lines. Therefore, we can identify

$$\mathcal{M} := \overline{\mathcal{M}}(F_{n-1}, F_0) = \mathcal{M}(F_{n-1}, F_0) \cong \{X \in SO(n) \mid X_{nn} = 0\}.$$

The groups $C_k(f)$ in the chain complex $C_\bullet(f), \partial$ are

$$C_k(f) = \begin{cases} C_k(F_0) & k < n-1 \\ C_k(F_0) \oplus C_{k-n+1}(F_{n-1}) & k \geq n-1 \end{cases}$$

and the complex can be pictured as follows:

$$\begin{tikzcd}
\dots \arrow[r, "\partial_0"] \arrow[ddr, "\partial_{n-1}"] & C_k(F_{n-1}) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_{n-1}"] & \dots \arrow[r, "\partial_0"] \arrow[ddr, "\partial_{n-1}"] & C_0(F_{n-1}) \arrow[r, "\partial_0"] \arrow[ddr, "\partial_{n-1}"] & 0 & \\
& \oplus & \dots & \oplus & \oplus & \\
\dots \arrow[r, "\partial_0"] & C_{k+n-1}(F_0) \arrow[r, "\partial_0"] \arrow[d, equal] & \dots \arrow[r, "\partial_0"] & C_{n-1}(F_0) \arrow[r, "\partial_0"] \arrow[d, equal] & C_{n-2}(F_0) \arrow[r, "\partial_0"] \arrow[d, equal] & \dots \\
\dots \arrow[r, "\partial"] & C_{k+n-1}(f) \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & C_{n-1}(f) \arrow[r, "\partial"] & C_{n-2}(f) \arrow[r, "\partial"] & \dots
\end{tikzcd}$$

The next result follows from the mapping cone.

Proposition 3.8 *If* $k < n-2$ *then*

$$H_k(SO(n)) \cong H_k(SO(n-1))$$

and if $k > \dim(SO(n-1)) + 1$ *then*

$$H_k(SO(n)) \cong H_{k-n+1}(SO(n-1))$$

Proof If $k < n-2$ then

$$H_{k-n+1}(F_{n-1}) = H_{k-n+2}(F_{n-1}) = 0.$$

Using Equation 3.3, we have the exact sequence

$$0 \to H_k(SO(n)) \to H_k(F_0) \to 0$$

and therefore $H_k(SO(n)) \cong H_k(F_0) \cong H_k(SO(n-1))$.

If $k > \dim(SO(n-1)) + 1$ then $H_k(F_0) = H_{k+1}(F_0) = 0$.Therefore, we have the exact sequence

$$0 \to H_{k-n+1}(F_{n-1}) \to H_k(SO(n)) \to 0$$

and so $H_k(SO(n)) \cong H_{k-n+1}(F_0) \cong H_{k-n+1}(SO(n-1))$. □

The next lemma gives an explicit formula for the flow of $X \in \mathcal{M}$:

Lemma 3.9 *Let* $X \in \mathcal{M}$. *Then*

$$\varphi_t(X)_{ij} = \begin{cases} -\tanh(\frac{t}{2}) & i = n \\ \frac{X_{ij}}{\cosh(\frac{t}{2})} & i = n, j \neq n \text{ or } j = n, i \neq n \\ X_{ij} - X_{in}X_{nj}\tanh\left(\frac{t}{2}\right) & i \neq n \neq j \end{cases}$$

Proof We are going to solve the ODE

$$\frac{d}{dt}\varphi_t(X) = -\nabla f(X)$$

as a flow of a vector field in $GL(n,\mathbb{R}) \supset SO(n)$.

Observe that

$$\frac{d}{dt}(\varphi_t(X)_{nn}) = \frac{1}{2}\left(\varphi_t(X)_{nn}^2 - 1\right),$$

which is independent of the other coordinates. Solving the ODE with the initial condition $\varphi_0(X)_{nn} = 0$ yields $\varphi_t(X)_{nn} = -\tanh\left(\frac{t}{2}\right)$.

The next thing is to find $\varphi_t(X)_{ij}$ where $i = n, j \neq n$ or $j = n, i \neq n$. The ODE in this case is

$$\frac{d}{dt}(\varphi_t(X)_{ij}) = -\frac{1}{2}\varphi_t(X)_{nn}\varphi_t(X)_{ij} = \frac{1}{2}\varphi_t(X)_{ij}\tanh\left(\frac{t}{2}\right)$$

with the initial condition $\varphi_0(X)_{ij} = X_{ij}$ and so the solution is

$$\varphi_t(X)_{ij} = \frac{X_{ij}}{\cosh\left(\frac{t}{2}\right)}.$$

If $i \neq n \neq j$, then

$$\frac{d}{dt}(\varphi_t(X)_{ij}) = \frac{1}{2} - \varphi_t(X)_{in}\varphi_t(X)_{nj} = -\frac{X_{in}X_{nj}}{2\cosh^2\left(\frac{t}{2}\right)}$$

and $\varphi_0(X)_{ij} = X_{ij}$. Integrating yields $\varphi_t(X)_{ij} = X_{ij} - X_{in}X_{nj}\tanh\left(\frac{t}{2}\right)$ $\square$

In particular, we have a very nice formula for the beginning map

$$e_- : \mathcal{M} \to F_{n-1}$$

from Theorem 2.5:

$$e_-(X) = \begin{pmatrix} X_{11} - X_{1n}X_{n1} & \dots & X_{1,n-1} - X_{1n}X_{n,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ X_{n-1,1} - X_{n-1,n}X_{n1} & \dots & X_{n-1,n-1} - X_{n-1,n}X_{n,n-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Using this explicit formula for the flow $\varphi_t(X)$, we can also characterize the moduli space $\mathcal{M} = \mathcal{M}(F_{n-1}, F_0)$.

Lemma 3.10 *The map $\psi : \mathcal{M} \to F_{n-1} \times S^{n-2}$ given by*

$$\psi(X) = (e_-(X), (X_{1n}, \dots, X_{n-1,n}))$$

is a diffeomorphism.

Proof First, e_- is smooth by Theorem 2.5 so ψ is smooth. Let

$$Y \in F_{n-1} \cong SO(n-1)$$

and $v = (v_1, \dots, v_{n-1}) \in \mathbb{R}^{n-1}$. We claim that ψ is a bijection and

$$\psi^{-1}(Y, v) = \begin{pmatrix} Y_{11} - v_1 w_1 & \dots & Y_{1,n-1} - v_1 w_{n-1} & v_1 \\ \vdots & \ddots & \vdots & \vdots \\ Y_{n-1,1} - v_{n-1} w_1 & \dots & Y_{n-1,n-1} - v_{n-1} w_{n-1} & v_{n-1} \\ -w_1 & \dots & -w_{n-1} & 0 \end{pmatrix}$$

where $w = v \cdot Y$ (i.e. $w_j = \sum_{k=1}^{n-2} Y_{kj} v_k$). Multiplying both sides by $Y^t = Y^{-1}$ yields $v = wY^t$ (so $v_i = \sum_{k=1}^{n-2} Y_{ik} w_k$).

Indeed, direct calculation gives that

$$(\psi \circ \psi^{-1})(Y, v) = (Y, v)$$

and

$$\psi^{-1}(\psi(X)) = X.$$

It remains to show that $X = \psi^{-1}(Y, v) \in SO(n)$. We first show that $XX^t = I$. First, since $v \in S^{n-2}$, $w = vY^t \in S^{n-2}$ and hence

$$(XX^t)_{nn} = \sum_{j=1}^{n-1} w_j^2 = 1.$$

If $i < n$, then

$$(XX^t)_{in} = (XX^t)_{ni} = \sum_{k=1}^{n-1} (Y_{ik} - v_i w_k) \cdot w_k = v_i - v_i = 0.$$

If $i \neq n \neq j$ then

$$\begin{aligned} (XX^t)_{ij} &= \sum_{k=1}^{n-1} (Y_{ik} - v_i w_k)(Y_{jk} - v_j w_k) + v_i v_j \\ &= \sum_{k=1}^{n-1} (Y_{ik} Y_{jk} - Y_{ik} w_k v_j - Y_{jk} w_k v_i + v_i v_j w_k^2) \\ &= \delta_{ij} - 2v_i v_j + 2v_i v_j = \delta_{ij}. \end{aligned}$$

Hence, $XX^t = I$ and so $X \in O(n)$. Since $\psi^{-1}(F_{n-1})$ is connected and

$$\psi^{-1}(F_{n-1}) \cap SO(n-1) \neq \emptyset,$$

we get that $X \in SO(n)$. Hence, ψ is a bijection $\mathcal{M} \to F_{n-1} \times S^{n-2}$. Since ψ and ψ^{-1} are smooth, ψ is a diffeomorphism. $\square$

Corollary 3.11 *For every $\sigma_P \in S_k^\infty(F_{n-1})$, the map $\psi_P : P \times_{F_{n-1}} \mathcal{M} \to P \times S^{n-2}$ given by*

$$\psi(x, X) = (x, (X_{1n}, \ldots, X_{n-1,n}))$$

is a diffeomorphism.

Proof Recall that

$$\begin{aligned} P \times_{F_{n-1}} \mathcal{M} &= \{(x, X) \in P \times \mathcal{M} \mid \sigma_P(x) = e_-(X)\} \\ &\cong \{(x, \psi(X)) \mid (x, X) \in P \times \mathcal{M},\ \sigma_P(x) = e_-(X)\}\,. \end{aligned}$$

Now, $\psi(X) = (e_-(X), (X_{1n}, \ldots, X_{n-1,n}))$ and $e_-(X)$ is uniquely determined by $\sigma_P(x)$. Therefore,

$$\begin{aligned} P \times_{F_{n-1}} \mathcal{M} &\cong \{(x, \psi(X)) \mid (x, X) \in P \times \mathcal{M},\ \sigma_P(x) = e_-(X)\} \\ &\cong \{(x, (X_{1n}, \ldots, X_{n-1,n})) \mid (x, X) \in P \times \mathcal{M},\ \sigma_P(x) = e_-(X)\}\,. \end{aligned}$$

□

We are now going to find a symmetry on $SO(n)$ that is invariant under the beginning and endpoint maps. More precisely, for every $X \in SO(n) \setminus \mathrm{Crit}(f)$ we find $X \neq \tilde{X} \in SO(n) \setminus \mathrm{Crit}(f)$ such that $\tilde{\tilde{X}} = X$ and

$$e_+(\tilde{X}) = e_+(X), \quad e_-(\tilde{X}) = e_-(X).$$

Definition 3.12 Let $X \in SO(n)$. We can write X as

$$X = \begin{pmatrix} U & v \\ w & x \end{pmatrix}$$

where $U \in M_{n-1}([-1, 1])$, $v^t, w \in [-1, 1]^{n-1}$, $x \in [-1, 1]$ and define

$$\tilde{X} := \begin{pmatrix} U & -v \\ -w & x \end{pmatrix}$$

Lemma 3.13 *Let $X \in SO(n)$. Then $\tilde{X} \in SO(n)$.*

Proof First, $\det(\tilde{X}) = \det(X) = 1$ since a row and a column are multiplied by -1. It remains to show that $\tilde{X}\tilde{X}^t = I$. Indeed,

$$(X\tilde{X}^t)_{ij} = \sum_{k=1}^{n} \tilde{X}_{ik}\tilde{X}_{jk} = \begin{cases} \sum_{k=1}^{n} X_{ik}X_{jk} = \delta_{ij} & i, j \neq n \text{ or } i = j = n, \\ -\sum_{k=1}^{n} X_{ik}X_{jk} = 0 & \text{else.} \end{cases}$$

□

Lemma 3.14 *$e_-(X) = e_-(\tilde{X})$ and $e_+(X) = e_+(\tilde{X})$.*

Proof Let φ_t be the flow of $-\nabla f$ and write

$$\varphi_t(X) = \begin{pmatrix} U_t & v_t \\ w_t & x_t \end{pmatrix}.$$

Since

$$\begin{aligned}
\nabla f(\tilde{X}) &= \frac{1}{2}\begin{pmatrix} -\tilde{X}_{1n}\tilde{X}_{n1} & \dots & -\tilde{X}_{1n}\tilde{X}_{n,n-1} & -\tilde{X}_{1n}\tilde{X}_{nn} \\ \vdots & \ddots & \vdots & \vdots \\ -\tilde{X}_{n-1,n}\tilde{X}_{n1} & \dots & -\tilde{X}_{n-1,n}\tilde{X}_{n,n-1} & -\tilde{X}_{n-1,n}\tilde{X}_{nn} \\ -\tilde{X}_{nn}\tilde{X}_{n1} & \dots & -\tilde{X}_{nn}\tilde{X}_{n,n-1} & 1-\tilde{X}_{nn}\tilde{X}_{nn} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -X_{1n}X_{n1} & \dots & -X_{1n}X_{n,n-1} & X_{1n}X_{nn} \\ \vdots & \ddots & \vdots & \vdots \\ -X_{n-1,n}X_{n1} & \dots & -X_{n-1,n}X_{n,n-1} & X_{n-1,n}X_{nn} \\ X_{nn}X_{n1} & \dots & X_{nn}X_{n,n-1} & 1-X_{nn}X_{nn} \end{pmatrix} = \widetilde{\nabla f(X)}
\end{aligned}$$

we get that $\varphi_t(\tilde{X}) = \widetilde{\varphi_t(X)}$. That is,

$$\varphi_t(\tilde{X}) = \begin{pmatrix} U_t & -v_t \\ -w_t & x_t \end{pmatrix}.$$

Using the fact that

$$\lim_{t\to\pm\infty} x_t = \mp 1,$$

we get that $\lim_{t\to\pm\infty} v_t = \lim_{t\to\pm\infty} w_t = 0$ and the result follows. □

Corollary 3.15 *Let $\sigma_P : P \to F_{n-1}$ be a singular C_p-space. Let*

$$(x, X) \in P \times_{F_{n-1}} \mathcal{M} = \{(x, X) \in P \times M \mid \sigma_P(x) = e_-(X)\}.$$

Then $(x, \tilde{X}) \in P \times_{F_{n-1}} \mathcal{M}$.

Let $X \in \mathcal{M}$, and write $\psi(X) = (Y, v) \in F_{n-1} \times S^{n-2}$. Then $\psi(\tilde{X}) = (Y, -v)$. This means that

$$\psi\left(\widetilde{\psi^{-1}(Y, v)}\right) = (Y, -v).$$

Using the above lemma, e_+ has a factorization

$$e_+ : \mathcal{M} \xrightarrow{\cong} F_{n-1} \times S^{n-2} \to F_{n-1} \times \mathbb{R}P^{n-2} \to F_0.$$

The next lemma provides a powerful tool to show that the map

$$(\partial_{n-1})_* : H_p(F_{n-1}) \to H_{p+n-2}(F_0)$$

is the zero map in some cases.

Lemma 3.16 *Let $\sigma_k : P_k \to F_{n-1}$ be singular C_p-spaces and let $\sigma := \sum_k n_k\sigma_k \in C_p(F_{n-1})$ be a cycle. Then there exists a p-dimensional CW-complex P', a smooth singular chain in $\sigma' \in S_{p+n-2}(P' \times S^{n-2})$ and a map*

$$\varphi : P' \times S^{n-2} \xrightarrow{(x,v)\mapsto(x,[v])} P' \times \mathbb{R}P^{n-2} \to F_0$$

so that

$$\partial_{n-1}(\sigma) - \varphi_*(\sigma') \in D^{\infty}_{p+n-2}(F_0).$$

Proof Let $\sigma_k : P_k \to F_{n-1}$ be singular C_p-spaces and let $\sum_k n_k\sigma_{P_k} \in S^{\infty}_p(F_{n-1})$ be a cycle. The process described in [2, Pages 108-109] can be applied analoguously for p-faces of I^N, giving a CW-complex P' consists of p-faces glued along parts of the boundary, a singular cycle $\sum_k n_k\sigma'_k \in S_p(P')$ and a continuous map $\xi : P' \to F_{n-1}$ such that

$$\sum_k n_k(\xi \circ \sigma'_k) = \sum_k n_k\sigma_k.$$

Note: Hatcher's book uses simplices for singular homology, so the process yields a p-dimensional simplicial complex. We use cubes, so applying the analoguous process yields a p-dimensional CW-complex.

Let $R := P' \times_{F_{n-1}} \mathcal{M} \cong P' \times S^{n-2}$ and define

$$\sigma_R : P' \times_{F_{n-1}} \mathcal{M} \xrightarrow{\pi_2} \mathcal{M} \xrightarrow{e_+} F_0.$$

For every k, write $R_k = P_k \times_{F_{n-1}} \mathcal{M}$. Then $\partial_{n-1}(\sigma_k) = \sigma_{R_k} : R_k \to F_0$ is the map

$$\sigma_R : P_k \times_{F_{n-1}} \mathcal{M} \xrightarrow{\pi_2} \mathcal{M} \xrightarrow{e_+} F_0.$$

By Lemma 2.36, for every k there is a smooth singular chain $\sum_{\alpha_k} n_{\alpha_k}\sigma_{\alpha_k} \in S_{p+n-2}(R)$ such that

$$\sigma_{R_k} - \sum_{\alpha_k} n_{\alpha_k}(\sigma_{R_k} \circ \sigma_{\alpha_k}) \in D^{\infty}_{p+n-2}(F_0).$$

Since $R_k \cong P_k \times S^{n-2}$ and $R = P' \times S^{n-2}$, one can identify $R_k \xhookrightarrow{(\sigma'_k \times id)} R$ and so $\sigma_R|_{R_k} = \sigma_{R_k}$. Therefore, we have a smooth singular chain

$$\sum_k \left(n_k \sum_{\alpha_k} n_{\alpha_k}\sigma_{\alpha_k} \right) \in S_{p+n-2}(R)$$

such that

$$\sum_k n_k\sigma_{R_k} - \sum_k \left(n_k \sum_{\alpha_k} n_{\alpha_k} \left(\sigma_R \circ \sigma_{\alpha_k}\right) \right) \in D^{\infty}_{p+n-2}(F_0)$$

which means

$$\partial_{n-1}\left(\sum_k n_k\sigma_k\right) - (\sigma_R)_*\left(\sum_k n_k \sum_{\alpha_k} n_{\alpha_k}\sigma_{\alpha_k}\right) \in D^{\infty}_{p+n-2}(F_0).$$

Now, because the map $X \mapsto \tilde{X}$ corresponds to the mapping to the map $(x,v) \mapsto (x,-v)$ in $R \cong P \times S^{n-2}$, and since $e_+(X) = e_+(\tilde{X})$,

$$\sigma_R(x,v) = \sigma_R(x,-v).$$

Hence, σ_R has a factorization

$$\sigma_R : P' \times S^{n-2} \to P' \times \mathbb{R}P^{n-2} \to F_0$$

and we complete the proof by taking $\varphi = \sigma_R$. □

Note: Although the Morse-Bott homology is defined only on compact oriented manifolds, the singular N-cube homology is defined on every manifold, and in particular, non-orientable manifolds.

3.3.1 Homology Groups Of $SO(2n)$

In this subsection we focus on the even case, which is easier using the fact that the map from S^{2n} to S^{2n} which sends $x \mapsto -x$ is orientation reversing.

Theorem 3.17 *For $0 \le k \le \frac{(2n-1)(2n-2)}{2}$, the map*

$$\partial_{2n-1} + \partial_{2n-1} : C_k(F_{2n-1}) \to C_{k+2n-2}(F_0)$$

is the zero map.

Proof Let $R := P \times_{F_{2n-1}} \mathcal{M} \in C_{k+2n-2}$ and let $\sigma_R : R \to F_0$ be the corresponding element in $C_{k+2n-2}(F_0)$. Since $R \cong P \times S^{2n-2}$ by Corollary 3.11, the map $\alpha : R \to R$ given by

$$\alpha(Y,v) = (Y,-v)$$

is an orientation-reversing diffeomorphism on R because the map $v \mapsto -v$ is orientation reversing on S^{2n-2}.

Let $X = \psi^{-1}(Y,v)$. Then $\tilde{X} = \psi^{-1}(Y,-v) = \psi^{-1}(\alpha(Y,v))$. Since

$$\sigma_R(X) = \sigma_R(\tilde{X}),$$

we get that $\sigma_R \circ \alpha = \sigma_R$.

Write $d(R) = \sum_j n_j R_j$. Then (using the notations of Definition 2.34)

$$\partial_0\sigma_R \circ \alpha = (-1)^{k+2n-2}\sum_j n_j \; (\sigma_R \circ \alpha)|_{\alpha^{-1}(R_j)} = (-1)^k \sum_j n_j R_j = \partial_0\sigma_R.$$

Therefore, $\partial_0\sigma_R \circ \alpha = \partial_0\sigma_R$. By condition (3) of degeneracy in Definition 2.34, we get that $\sigma_R + \sigma_R \in D_p^\infty(F_0)$ and thus

$$(\partial_{2n-1} + \partial_{2n-1})(P) = \sigma_R + \sigma_R \in D_p^\infty(F_0).$$

Since $C_{k+2n-1}(F_0) = S^\infty_{k+2n-2}(F_0)/D^\infty_{k+2n-2}(F_0)$ we get the result. □

In addition, using Lemma 3.16 and the long exact sequence in 3.3, we can get a short exact sequence on homology in the even case:

Theorem 3.18 *For every $k \in \mathbb{N}$ there is a short exact sequence*

$$0 \to H_k(SO(2n-1)) \xrightarrow{i_*} H_k(SO(2n)) \xrightarrow{j_*} H_{k-2n+1}(SO(2n-1)) \to 0 \quad (3.4)$$

where i_* and j_* are the maps from the mapping cone.

Proof Let $\sigma \in C_p(F_{2n-1})$ be a cycle. By Lemma 3.16, there is a p-dimensional CW-complex P', a map

$$\varphi : P' \times S^{2n-2} \to P \times \mathbb{R}P^{2n-2} \to F_0$$

and a smooth singular chain $\sigma' \in S_{p+2n-2}(P' \times S^{2n-2})$ such that

$$\partial_{2n-1}(\sigma) - \varphi_*(\sigma') \in D^\infty_{p+2n-2}(F_0).$$

By Künneth formula [2, Theorem 3B.6] there is a short exact sequence

$$\begin{aligned} 0 \to \bigoplus_i H_i(P') \otimes H_{p+2n-2-i}(\mathbb{R}P^{2n-2}) &\to H_{p+2n-2}(P' \times \mathbb{R}P^{2n-2}) \\ &\to \bigoplus_i \operatorname{Tor}(H_i(P'), H_{p+2n-3-i}(\mathbb{R}P^{2n-2})) \to 0. \end{aligned}$$

Now, since P' is a p-dimensional CW-complex, $H_k(P') = 0$ for $k > p$. In addition, $H_k(\mathbb{R}P^{2n-2}) = 0$ for every $k \geq 2n-2$, as it is a non-orientable $(2n-2)$-dimensional manifold. Therefore $H_i(P') \otimes H_{p+2n-2-i}(\mathbb{R}P^{2n-2}) = 0$ for every i. In addition, the only possible i for which both $H_i(P') \neq 0$ and $H_{p+2n-3-i}(\mathbb{R}P^{2n-2}) \neq 0$ is $i = p$.

Moreover, $H_p(P')$ has no torsion because there are no $(p+1)$-cells, and hence ([2, Proposition 3A.5]),

$$\operatorname{Tor}(H_p(P), H_{2n-3}(\mathbb{R}P^{2n-2})) = 0$$

Thus, $H_{p+n-2}(P \times \mathbb{R}P^{2n-2}) = 0$. The map $\varphi : P' \times S^{n-2} \to F_0$ induces a map on homology

$$\varphi_* : H_{p+2n-2}(P' \times S^{n-2}) \to H_{p+2n-2}(P' \times \mathbb{R}P^{n-2}) \to H_{p+2n-2}(F_0).$$

Since $H_{p+2n-2}(P' \times \mathbb{R}P^{n-2}) = 0$, φ_* is the zero map. Therefore,

$$\varphi_* \left([\sigma'] \right) = 0 \in H_{p+2n-2}(F_0)$$

and since

$$\partial_{2n-1}(\sigma) - \varphi_*(\sigma') \in D^{\infty}_{p+2n-2}(F_0),$$

also $[\partial_{2n-1}(\sigma)] = 0 \in H_{p+2n-2}(F_0)$. Hence,

$$(\partial_{2n-1})_* : H_p(F_{2n-1}) \to H_{p+2n-2}(F_0)$$

is the zero map.

Recall the long exact sequence 3.3:

$$\cdots \xrightarrow{(\partial_{2n-1})_*} H_k(F_0) \xrightarrow{i_*} H_k(M) \xrightarrow{j_*} H_{k-2n+1}(F_{2n-1}) \xrightarrow{(\partial_{2n-1})_*} H_{k-1}(F_0) \to \cdots$$

Then $(\partial_{2n-1})_* = 0$. This means that

$$i_* : H_k(SO(2n-1)) \to H_k(SO(2n))$$

is injective and

$$j_* : H_k(SO(2n)) \to H_{k-2n+1}(SO(n-1))$$

is surjective. Hence, the sequence

$$0 \to H_k(F_0) \xrightarrow{i_*} H_k(M) \xrightarrow{j_*} H_{k-2n+1}(F_{2n-1}) \to 0$$

is exact. Since $F_0 \cong F_{2n-1} \cong SO(2n-1)$ and $M = SO(2n)$, the result follows. $\square$

Corollary 3.19 *If $H_{k-2n+1}(SO(2n-1))$ is free, then*

$$H_k(SO(2n)) \cong H_k(SO(2n-1)) \oplus H_{k-2n+1}(SO(2n-1)).$$

In particular,

$$H_{2n-1}(SO(2n)) \cong H_{2n-1}(SO(2n-1)) \oplus \mathbb{Z}.$$

We need the following simple lemma for the corollary:

Lemma 3.20 *Let*

$$0 \to A \to B \xrightarrow{j} C \to 0$$

be a short exact sequence of R-modules. Assume that C is a free R-module. Then the sequence splits, that is, $B = A \oplus C$.

Proof Let $\{c_i\}_{i\in I}$ be a basis for C. Define $s : C \to B$ by $s(c_i) = b_i \in j^{-1}(c_i)$ ($j^{-1}(c_i)$ is not empty because j is surjective) for every $i \in I$ and extend linearly. Then $j \circ s : C \to C$ is the identity map, so the sequence splits by [2, Splitting Lemma]. □

Proof (proof of 3.19) If $H_{k-2n+1}(SO(n-1))$ is free then the SES splits.

In addition, $H_0(SO(2n-1)) \cong \mathbb{Z}$ because $SO(2n-1)$ is connected. Hence,

$$H_{2n-1}(SO(2n)) \cong H_{2n-1}(SO(2n-1)) \oplus \mathbb{Z}.$$ □

Example 3.21 Let $M = SO(4)$. Since $SO(3) \cong \mathbb{R}P^3$ [2, Pages 293-294], its homology groups are:

$$H_k(SO(3)) = \begin{cases} \mathbb{Z} & k = 0, 3 \\ \mathbb{Z}_2 & k = 1 \\ 0 & \text{else} \end{cases}.$$

By Proposition 3.8, $H_k(SO(4)) \cong H_k(SO(3))$ for $k \leq 2$ and $H_k(SO(4)) \cong H_{k-3}(SO(3))$ for $k \geq 4$. Therefore, the only remaining case is $k = 3$. By Theorem 3.18, there is a short exact sequence

$$0 \to H_3(SO(3)) \xrightarrow{i_*} H_3(SO(4)) \xrightarrow{j_*} H_0(SO(3)) \to 0$$

and since $H_0(SO(3)) \cong H_3(SO(3)) \cong \mathbb{Z}$, $H_0(SO(3))$ is free and by Corollary 3.19,

$$H_3(SO(4)) \cong H_3(SO(3)) \oplus H_0(SO(3)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Therefore, the homology groups of $SO(4)$ are:

$$H_k(SO(4)) = \begin{cases} \mathbb{Z} & k = 0, 6 \\ \mathbb{Z}_2 & k = 1, 4 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 3 \\ 0 & \text{else} \end{cases}.$$

3.3.2 Homology Over $\mathbb{Z}_2$

The long exact sequence of homology in 3.3 can be obtained the same way also over $\mathbb{Z}_2$:

$$\cdots \xrightarrow{(\partial_{n-1})_*} H_k(F_0; \mathbb{Z}_2) \xrightarrow{i_*} H_k(M; \mathbb{Z}_2)$$
$$\xrightarrow{j_*} H_{k-n}(F_n; \mathbb{Z}_2) \xrightarrow{(\partial_{n-1})_*} H_{k-1}(F_0; \mathbb{Z}_2) \to \cdots.$$

In our case, $M = SO(n)$ and $F_0 \cong F_{n-1} \cong SO(n-1)$. Hence, we can write the following exact sequence as follows:

$$\cdots \xrightarrow{(\partial_{n-1})_*} H_k(SO(n-1);\mathbb{Z}_2) \xrightarrow{i_*} H_k(SO(n);\mathbb{Z}_2)$$
$$\xrightarrow{j_*} H_{k-n}(F_n;\mathbb{Z}_2) \xrightarrow{(\partial_{n-1})_*} H_{k-1}(SO(n-1);\mathbb{Z}_2) \to \cdots .$$

Similarly to Theorem 3.18, we can derive an SES for homology over $\mathbb{Z}_2$:

Theorem 3.22 *For every $k \in \mathbb{N}$ there is a short exact sequence*

$$0 \to H_k(SO(n-1);\mathbb{Z}_2) \xrightarrow{i_*} H_k(SO(n);\mathbb{Z}_2) \xrightarrow{j_*} H_{k-n+1}(SO(n-1);Z_2) \to 0 \tag{3.5}$$

where i_* and j_* are the maps from the mapping cone.

Proof Let $\sigma \in C_p(F_{n-1})$ be a cycle. By Lemma 3.16, there is a p-dimensional CW-complex P', a map

$$\varphi : P' \times S^{n-2} \xrightarrow{(x,v)\mapsto(x,[v])} P \times \mathbb{R}P^{n-2} \to F_0$$

and a smooth singular chain $\sigma' \in S_{p+n-2}(P' \times S^{n-2})$ such that

$$\partial_{n-1}(\sigma) - \varphi_*(\sigma') \in D^{\infty}_{p+n-2}(F_0).$$

By Künneth formula for coefficients in a field [2, Corollary 3B.7], there is an isomorphism

$$h : \bigoplus_i H_i(P';\mathbb{Z}_2) \otimes H_{p+n-2-i}(S^{n-2};\mathbb{Z}_2) \to H_{p+n-2}(P' \times S^{n-2};\mathbb{Z}_2).$$

Since

$$H_k(P';\mathbb{Z}_2) = 0$$

for every $k > p$ and

$$H_k(S^{n-2};\mathbb{Z}_2) = 0$$

for every $k > n-2$, we get that

$$H_p(P';\mathbb{Z}_2) \otimes H_{n-2}(S^{n-2};\mathbb{Z}_2) \cong H_{p+n-2}(P' \times S^{n-2};\mathbb{Z}_2).$$

Similarly,

$$H_p(P';\mathbb{Z}_2) \otimes H_{n-2}(\mathbb{R}P^{n-2};\mathbb{Z}_2) \cong H_{p+n-2}(P' \times \mathbb{R}P^{n-2};\mathbb{Z}_2).$$

The map $\xi : S^{n-2} \to \mathbb{R}P^{n-2}$ defined by $\xi(v) = [v]$ has degree 0 mod 2, and hence

$$\xi_* : H_{n-2}(S^{n-2}) \to H_{n-2}(\mathbb{R}P^{n-2})$$

is the zero map. Using the isomorphisms from Künneth formula, the map

$$(id \times \xi)_* : H_p(P';\mathbb{Z}_2) \otimes H_{n-2}(S^{n-2};\mathbb{Z}_2) \to H_p(P';\mathbb{Z}_2) \otimes H_{n-2}(\mathbb{R}P^{n-2};\mathbb{Z}_2)$$

is also the zero map. Since φ_* can be written as $\varphi_* = (id \times \xi)_* \varphi'_*$, φ_* is the zero map as well.

Since $\partial_{n-1}(\sigma) - \varphi_*(\sigma') \in D^{\infty}_{p+n-2}(F_0)$,

$$(\partial_{n-1})_* (\sigma) = \varphi_*(\sigma') = 0 \in H_{p+n-2}(F_0;\mathbb{Z}_2)$$

so $(\partial_{n-1})_* : H_p(F_{n-1};\mathbb{Z}_2) \to H_{p+n-2}(F_0,\mathbb{Z}_2)$ is the zero map.

We have the long exact sequence

$$\begin{aligned} \cdots \xrightarrow{(\partial_{n-1})_*} H_k(SO(n-1);\mathbb{Z}_2) &\xrightarrow{i_*} H_k(SO(n);\mathbb{Z}_2) \\ &\xrightarrow{j_*} H_{k-n}(F_n;\mathbb{Z}_2) \xrightarrow{(\partial_{n-1})_*} H_{k-1}(SO(n-1);\mathbb{Z}_2) \to \cdots . \end{aligned}$$

Therefore, the map

$$i_* : H_k(SO(n-1)) \to H_k(SO(n))$$

is injective and the map

$$j_* : H_k(SO(n)) \to H_{k-n+1}(SO(n-1))$$

is surjective. Hence, the sequence

$$0 \to H_k(SO(n-1);\mathbb{Z}_2) \xrightarrow{i_*} H_k(SO(n);\mathbb{Z}_2) \xrightarrow{j_*} H_{k-n+1}(SO(n-1);Z_2) \to 0$$

is exact. □

Now, $H_{k-n+1}(SO(n-1);Z_2)$ is a vector field over $\mathbb{Z}_2$, so it is a free $\mathbb{Z}_2$-module. Using Lemma 3.20, we get:

Theorem 3.23

$$H_k(SO(n);\mathbb{Z}_2) \cong H_k(SO(n-1);\mathbb{Z}_2) \oplus H_{k-n+1}(SO(n-1);\mathbb{Z}_2).$$

This theorem is already known. The proof here is very similar to the one in [8, Section 5], where they used the same Morse-Bott function but with different system. There is a proof using cell structure in Hatcher's book [2, Theorem 3D.1], and it can be also derived from 3.3 using [18, Theorem 3].

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